1 Lecture 1

1.1 Two-Distance Sets

Definition 1.1. A two-distance set in $\mathbb{R}^n$ is a set of points $\{p_1, \ldots, p_m\}$ such that $\text{dist}(p_i, p_j) \in \{\alpha, \beta\}$ for $i \neq j$. That is, the distance between any two distinct points in the set has one of two fixed values.

Some examples in the plane: an isosceles triangle, a square, a regular pentagon.

Exercise 1.2. Prove: In a regular pentagon, the ratio of a diagonal to a side is $\frac{1 + \sqrt{5}}{2}$, the golden ratio.

Exercise 1.3. Prove that a two-distance set in $\mathbb{R}^2$ has at most 5 points.

Similarly, we can define one-distance sets. In $\mathbb{R}^2$, an equilateral triangle is the only example of a one-distance set with three points. In $\mathbb{R}^3$, a regular tetrahedron is a one-distance set with four points. Similarly, the $n$-dimensional regular simplex is a one-distance set with $n + 1$ points in $\mathbb{R}^n$.

It is not immediate how to give coordinates for the regular simplex (an equilateral triangle with one side on the axis will have coordinates $(0, 0), (1, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, but there is a nice way to give its coordinates in $\mathbb{R}^{n+1}$. Take the standard basis vectors in $\mathbb{R}^{n+1}$, which have coordinates $(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$. These all lie on the $n$-dimensional hyperplane given by the equation $x_1 + x_2 + \cdots + x_n = 1$, which is isometric to $\mathbb{R}^n$. And the distance between any two of these points is $\sqrt{2}$, so this is in fact the $n$-dimensional regular simplex.

Now, we ask how many points a two-distance set in $\mathbb{R}^n$ can have. We will denote this maximum number by $m_2(n)$.

We can get $n + 2$ points by reflecting one vertex of the $n$-dimensional simplex across the plane of the rest of the vertices. We can get $2n$ by taking points of the form $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$ in $\mathbb{R}^n$. This will give us distances 2 and $\sqrt{2}$. This gives us a square in 2 dimensions, and in general it gives us the $n$-dimensional (hyper)octahedron.
Exercise 1.4. Find a two-distance set in \( \mathbb{R}^n \) with a quadratic number of points.

Theorem 1.5. \( m_2(n) \leq \frac{(n+1)(n+4)}{2} \sim n^2 \), where \( \sim \) refers to asymptotic equality.

This theorem is asymptotically best possible, as a simple solution to the preceding exercise will show.

Proof. Let \( \{x_1, \ldots, x_m\} \subset \mathbb{R}^n \) be a two-distance set. Let \( \alpha, \beta \) be the two distances.

The distance between any two points \( x \) and \( y \) is the norm of their difference, i.e.,

\[
\text{dist}(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.
\]

Note that, for fixed \( y \), \( \text{dist}(y, z)^2 = \sum_{i=1}^{n} (y_i - z_i)^2 \) is a polynomial in the coordinates of \( z \).

Also, note that \( \text{dist}(x, y) = \alpha \iff \text{dist}(x, y)^2 - \alpha^2 = 0 \).

Now, for \( t \in \mathbb{R}^n \) let \( f_i(t) = (\text{dist}(x_i, t)^2 - \alpha^2)(\text{dist}(x_i, t)^2 - \beta^2) \). This is a polynomial in \( n \) variables. It will be zero when the distance between \( x_i \) and \( t \) is either \( \alpha \) or \( \beta \). Therefore, if the \( x_i \) form a two-distance set, we have

\[
f_i(x_j) = \begin{cases} \alpha^2 \beta^2 \neq 0 & i = j \\ 0 & i \neq j \end{cases}
\]

Claim 1.6. The polynomials \( f_i(t) \in \mathbb{R}[t_1, \ldots, t_n] \) are linearly independent over \( \mathbb{R} \).

Proof. Suppose that some linear combination of the \( f_i \) is zero:

\[
\sum_{i=1}^{m} \gamma_i f_i = 0 \quad (1.1)
\]

We wish to show that each \( \gamma_i \) is zero.

We substitute \( x_j \) in this expression, and we get \( \sum_{i=1}^{m} \gamma_i f_i(x_j) = \gamma_j \alpha^2 \beta^2 = 0 \). \( \alpha^2 \beta^2 \) is nonzero, so we must have \( \gamma_j = 0 \) for all \( j \). Therefore, the \( f_i \) are linearly independent, and the claim is proved.

Now, we want to prove that the \( f_i \) reside in some low-dimensional space, so we can take advantage of the following basic fact of linear algebra:

Fact 1.7. If \( f_1, \ldots, f_m \) are linearly independent elements of a vector space, and each one is a linear combination of some elements \( g_1, \ldots, g_k \), then \( m \leq k \).

So, it remains to find a small set of polynomials in terms of which all the \( f_i \) can be expressed.
Theorem 1.10 (Bollobás).

Let the sets $A$ and $B = \{A_1, \ldots, A_m\}$, where for all $1 \leq i, j \leq m$, $|A_i| = r, |B_i| = s$, and $|A_i \cap B_j| = \begin{cases} 0 & i = j \\ \neq 0 & i \neq j \end{cases}$.

Let $m(r, s)$ be the maximum $m$ for a set-system satisfying these requirements.

For $s = 1$, we can achieve $m = r + 1$, by taking $X = \{p_1, \ldots, p_{r+1}\}$ to be a set with $r + 1$ elements, $B_i = \{p_i\}$, and $A_i = X \setminus B_i$. Therefore, $m(r, 1) \geq r + 1$. To prove that $m(r, 1) \leq r + 1$, note that each $B_i$ is a single point, so that $A_i \supseteq B_2 \cup \cdots \cup B_m$. Therefore, $A_1$ (and all other $A_i$) must contain $m - 1$ elements, so $r = m - 1$.

$m(r, 2) \geq \binom{r+2}{2} = \binom{r+2}{r}$. Let the $B_i$ be all the pairs from a set $X$ with $r + 2$ elements, and let $A_i = X \setminus B_i$. Similarly, we get $m(r, s) \geq \binom{r+s}{s}$ by taking a universe of $r + s$ elements, letting the $B_i$ be all sets of size $s$, and letting $A_i = X \setminus B_i$.

Theorem 1.10 (Bollobás). (abridged) $m(r, s) = \binom{r+s}{s}$

The full version of this theorem will be stated next time.