REU 2005 · Potpourri · Lecture 2

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2 Sperner's Theorem and Bollobás' Inequality

2.1 Sperner's Theorem

Definition 2.1 (Sperner Family). Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ be a family of subsets of $H := \{1, 2, \ldots, n\}$. $\mathcal{F}$ is called a Sperner Family if $\forall i \neq j$, $A_i \not\subset A_j$.

Example 2.2 (Singletons). *For example the family of singletons $\mathcal{F} = \{\{1\}, \{2\}, \ldots, \{n\}\}$ is a Sperner Family. In fact it is a is* maximal *Sperner Family. By* maximal *we mean that we cannot add another set and keep the family Sperner.*

Definition 2.3 (k-uniform). A family of subsets $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ of $H = \{1, 2, \ldots, n\}$ is called a k-uniform family if $\forall j, |A_j| = k$.

Example 2.4 (k-subsets). *If $X =:= \{1, 2, \ldots, n\}$, let $\binom{X}{k} := \{A \mid A \subset X, |A| = k\}$. That is $\binom{X}{k}$ is set of all subsets of X os size k. For all k, the family $\binom{X}{k}$ is a maximal Sperner family. Note also that $|\binom{X}{k}| = \binom{n}{k}$.*

Notice that $\binom{H}{k}$ is a k-uniform family.

Question 2.5. *What is the* maximum *size of a Sperner Family?*

For $k = \lfloor \frac{n}{2} \rfloor$, the family $\binom{H}{\lfloor \frac{n}{2} \rfloor}$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Sperner in 1928 proved that every Sperner family has size at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Theorem 2.6 (Sperner's Theorem). *Let $A_1, \cdots, A_m$ be a Sperner family of subsets of a set of n elements. Then*

$$m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

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We need some notions from finite probability spaces before we get to the proof.

2.2 Crash Course on Probability

Definition 2.7. A finite probability space is a non-empty finite set Ω together with a function $\Pr : \Omega \to \mathbb{R}$ such that

1. $(\forall x \in \Omega)(\Pr(x) > 0)$
2. $\sum_{x \in \Omega} \Pr(x) = 1$

The set Ω is the sample space and the function Pr is the probability distribution. The elements ω are called atomic events or elementary events. An event is a subset of Ω. For $A \subseteq \Omega$, we define the probability of A to be $\Pr(A) := \sum_{\omega \in A} \Pr(A)$: In particular, for atomic events we have $\Pr(\{\omega\}) = \Pr(\omega)$; and $\Pr(\emptyset) = 0$, $\Pr(\Omega) = 1$.

The uniform distribution over the sample space Ω is defined by setting $\Pr(\omega) = 1/|\Omega|$ for every $\omega \in \Omega$.

Exercise 2.8 (Union Bound). *Let* $(\Omega, \Pr)$ *be a probability space and* $A_1, A_2, \dots, A_m$ *be events. then*

$$\Pr(\bigcup_{i=1}^{m} A_i \leq \sum_{i=1}^{m} \Pr(A_i)$$

Definition 2.9 (Random Variable). A random variable is a function $X : \Omega \to \mathbb{R}$.

We say that X is constant if $X(\omega)$ takes the same value for all $\omega \in \Omega$.

Definition 2.10. The expected value of a random variable X is $\mathrm{E}(X) = \sum_{\omega \in \Omega} X(\omega)\Pr(\Omega)$.

Exercise 2.11. $\min X \leq \mathrm{E}(X) \leq \max X$

Exercise 2.12 (Linearity of Expectation). *If* $X_1, X_2, \dots, X_k$ *be random variables. Prove that*

$$\mathrm{E}(X_1 + \cdots + X_k) = \sum_{i=1}^{k} \mathrm{E}(X_i)$$

Definition 2.13. The indicator variable of an event $A \subset \Omega$ is the function $\vartheta : \Omega \to \{0, 1\}$ given by

$$\vartheta_A(\omega) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

Exercise 2.14. *The expected value of an indicator variable* ϑ_A *is* $\mathrm{E}(\vartheta_A) = \Pr(A)$.

Example 2.15. *A club with 2005 members distributes membership cards numbered 1 through 2005 to its members at random. Members whose card number happens to coincide with their year of birth receive a prize. Determine the expected number of lucky members. Surprisingly the answer will not depend on the age distribution of the club members.*

Proof. First we observe that we have the unifrom distribution over a sample space of size 2005!. Let X be number of lucky members if the membership cards are distributed at random. Define the indicator variables

$$\vartheta_i = \begin{cases} 1 & \text{if the } i^{th} \text{ (in alphabetical order) member is lucky} \\ 0 & \text{if the } i^{th} \text{ member is not lucky} \end{cases}$$

Notice that

$$X = \vartheta_1 + \vartheta_2 + \ldots + \vartheta_{2005}.$$

Hence

$$\mathrm{E}(\psi) = \mathrm{E}(\vartheta_1) + \mathrm{E}(\vartheta_2) + \ldots + \mathrm{E}(\vartheta_{2005}).$$

But $\mathrm{E}(\vartheta_i) = \Pr(i^{th} \text{ member is lucky}) = \frac{1}{2005}$ since the i^{th} has equal chance of getting any card. Hence $\mathrm{E}(X) = 1$. □

2.3 Proof of Sperner's Theorem

Now we prove the Sperner's Theorem. The proof we present is due to Lubell (1966). Our proof is through a result customarily referred to as the LYM inequality (L=Lubell, Y= Yamamoto,M=Meshalkin). We call it BLYM-inequality to honor Bollobás who produced a more general inequality earlier(see below).

Theorem 2.16 (BLYM Inequality). *If* $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ *is a Sperner family of* $X = \{1, 2 \ldots, n\}$*, then*

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1.$$

Proof of Sperner's Theorem using (B)LYM inequality: Since for all $1 \leq i \leq m$,

$$\binom{n}{|A_i|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

we get that

$$1 \geq \sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \geq \sum_{i=1}^{m} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \frac{m}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

□

Before we prove the LYM-inequality we need the notion of chains.

Definition 2.17 (Chain of subsets). A family of subsets $\mathcal{B} = \{B_1, B_2 \ldots, B_m\}$ is called a chain if $\forall(i, j)(B_i \subseteq B_j$ or $B_j \subseteq B_i)$. A maximal chain is a chain of *distinct* subsets which cannot be expanded by adding another subset. Maximal chains increase in increments of 1.

There is a natural one-to-one correspondence between the set of permutations of X and the set of maximal chains over X. Given a permutation $\sigma : X \to X$, the chain defined by $B_0 = \emptyset$, $B_i = \{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ is maximal. It is easy to see that this function is one-to-one.

Proof of Theorem 2.16: This proof technique is called the *Lubell's permutation method.* Consider a random maximal chain of subsets of X

$$\mathcal{B} = \{B_0, \ldots, B_n\}$$

(By random we mean that the sample space is the set of all maximal chains and the probability distribution is uniform). Define the random variable X as

$$X = |\mathcal{F} \cap \mathcal{B}|$$

Notice that $X \leq 1$. This is because otherwise there are two sets A_i and A_j which occur in both $\mathcal{F}$ and $\mathcal{B}$ but these two requirements are contradictory (one requires them to be incomparable and the other requires them to comparable.) Hence

$$\mathrm{E}(X) \leq 1.$$

Now define the indicator variables ϑ_i as

$$\vartheta_i = \begin{cases} 1 & \text{if } A_i \in \mathcal{B} \\ 0 & \text{if } A_i \notin \mathcal{B} \end{cases}.$$

Notice that $X = \vartheta_1 + \vartheta_2 + \cdots + \vartheta_m$. Hence $\mathrm{E}(X) = \mathrm{E}(\vartheta_1) + \mathrm{E}(\vartheta_2) + \cdots + \mathrm{E}(\vartheta_m)$.

$$\begin{aligned} \mathrm{E}(\vartheta_i) &= \Pr(A_i \in \mathcal{B}) && (2.1) \\ &= \frac{1}{\binom{n}{|A_i|}}. && (2.2) \end{aligned}$$

The reason of equation 2.1 is the following. Let $k = |A_i|$. Then the set $B_k \in \mathcal{B}$ can be chose in $\binom{n}{k}$ ways, and all have the equal probability. So $\Pr(B_k = A_i) = \frac{1}{\binom{n}{|A_i|}}$. □

2.4 Bollobás's Inequality

We had seen the following version of the Bollobás's theorem earlier

Theorem 2.18. *[Bollobás's inequality, uniform version] If $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ is a r-uniform family and $\mathcal{G} = \{B_1, B_2, \ldots, B_m\}$ is a s-uniform family such that $(\forall i)(A_i \cap B_i = \emptyset)$ and $(\forall i \neq j)(A_i \cap B_j \neq \emptyset)$ then $m \leq \binom{r+s}{r}$.*

Bollobás's main theorem refers to pairs of not necessarily uniform set systems.

Theorem 2.19 (Bollobás' inequality, general case). *If $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{G} = \{B_1, B_2, \ldots, B_m\}$ are two families such that $\forall(i)(A_i \cap B_i = \emptyset)$ and $\forall(i \neq j)(A_i \cap B_j \neq \emptyset)$ then*

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$$

Exercise 2.20. *Prove Theorem 2.18 using Theorem 2.19*

Exercise 2.21. *Prove that Bollobás's inequality (Theorem 2.19) implies the BLYM inequality.*

Exercise 2.22. *Generalize Lubell's proof to prove Bollobás's theorem(Theorem refthm-boll)*

The following strengthening of the uniform version of Bollobás's theorem called the *Skew Bollobás theorem* was proved by Lovász.

Theorem 2.23 (("Skew Bollobás Theorem")Lovász). *If $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ is a r-uniform family and $\mathcal{G} = \{B_1, B_2, \ldots, B_m\}$ is a s-uniform family such that $(\forall i)(A_i \cap B_i = \emptyset)$ and $(\forall i < j)(A_i \cap B_j \neq \emptyset)$ then $m \leq \binom{r+s}{r}$.*

Note that we dropped nearly half the constraints: no assumption is made on $A_i \cap B_j$ if $i > j$

Exercise 2.24.** *Prove the Skew Bollobás Theorem.*
*(*Hint*: Linear algebra method. determinants.)*