3 Spectral Properties of Graphs

3.1 2-Distance set
Recall the definition of 2-distance sets in $\mathbb{R}^n$:

**Definition 3.1.** A 2-distance set in $\mathbb{R}^n$ is a set of points $\{p_1, \ldots, p_m\}$ such that $\operatorname{dist}(p_i, p_j) \in \{\alpha, \beta\}$ for $i \neq j$. That is, the distance between any two distinct points in the set has one of two fixed values.

We denote by $m_2(n)$ the maximum size of a 2-distance set in $\mathbb{R}^n$. 

**Exercise 3.2.** Prove that $m_2(n) > cn^2$ for some constant $c$.

Recall that a one-distance set in $\mathbb{R}^n$ has $\leq (n+1)$ vertices. Recall that the set of standard unit vectors $\{e_i : 1 \leq i \leq n\}$ is a set of size $n$ and is a one-distance set in $\mathbb{R}^n$.

Taking clue from the above example we can say that the set $\{(e_i + e_j) : 1 \leq i < j \leq n\}$ is a 2-distance set in $\mathbb{R}^n$. The size of this set is $\binom{n}{2}$.

**Exercise 3.3.** $m_2(n) \geq \binom{n+1}{2}$.

3.2 Linear Algebra
Consult the Linear Algebra and applications to graphs Part I and II handouts.

3.3 Adjacency matrix of graph
Let $G$ be a simple graph (that is, no multiple edges and no self loop). Let the adjacency matrix is $A$. And let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of $G$.

In other words the characteristic polynomial $f_A(x)$ factors as $f_A(x) = \Pi(x - \lambda_i)$.

**Exercise 3.4.** Prove that $\deg_{\min} \leq \lambda_1 \leq \deg_{\max}$, where $\deg_{\min}$ and $\deg_{\max}$ are the minimum and maximum degrees of the graph, respectively.
Exercise 3.5. \( \lambda_1 \geq \) average degree. Here “average degree” is \( \frac{\sum_{i \in V} \text{deg}(i)}{n} = \frac{2m}{n} \), where \( m \) is number of edges.

Exercise 3.6. \( \sum_{i=1}^{n} \lambda_i = 0 \).

By the trace of a matrix we mean the sum of the diagonal entries.

Exercise 3.7. For any square matrix the sum of the eigenvalues is same as the trace.

From this it follows that unless \( G \) is the empty graph (no edges) \( \lambda_1 > 0 \) and \( \lambda_n < 0 \).

Exercise 3.8. Prove that \( |\lambda_n| \leq \lambda_1 \).

Exercise 3.9. If \( G \) is bipartite then prove that \( \lambda_{n-i} = -\lambda_{i+1} \) for all \( 0 \leq i < n \).

Conversely,

Exercise 3.10. Prove that if \( G \) is connected and \( \lambda_n = -\lambda_1 \) then \( G \) is bipartite. Show that this statement fails if we drop the assumption of connectedness.

Exercise 3.11. If \( M \) is a square matrix over \( \mathbb{C} \) with eigenvalues \( \mu_1, \mu_2, \ldots, \mu_n \), then the eigenvalues of \( M^2 \) are \( \mu_1^2, \mu_2^2, \ldots, \mu_n^2 \).

The proof of the above exercise is easier when \( M \) is real symmetric.

Corollary 3.12. \( \sum_{i=1}^{n} \mu_i^2 = \text{trace of } M^2 \).

Exercise 3.13. Give a combinatorial meaning to \( \sum_{i=1}^{n} \lambda_i^2 \).

[HINT (1) : The answer is very simple]

[HINT (2) : The above corollary]

Exercise 3.14. Find the combinatorial meaning of each entry of \( A^2 \).

Definition 3.15. A regular graph of degree \( r \) is a graph for which the degree of all the vertices is \( r \).

From a previous exercise we get that for a regular graph \( \lambda_i = r \).

Exercise 3.16. Find a eigenvector corresponding to eigenvalue \( r \) of the regular graph of degree \( r \).

Exercise 3.17. What is the multiplicity of the eigenvalue \( r \)? [HINT: Answer is a very simple combinatorial parameter of \( G \).]
3.4 Interlacing

Exercise* 3.18. Let $G$ be a general graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $H = G \setminus v$ (that is, the graph $G$ with the vertex $v$ deleted) and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of $H$. Then,

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$$

This property is called interlacing.

We can actually prove a more general statement.

Exercise* 3.19. Let $A$ be a $n \times n$ real symmetric matrix. Let $B$ is the matrix obtained by deleting the $i$-th row and $i$-th column of $A$. Prove that the eigenvalues of $B$ interlace the eigenvalues of $A$.

[HINT (1): Use Spectral Theorem]
[HINT (2): Prove $\lambda_1 \geq \mu_1$ using the following exercise]

Exercise 3.20 (Courant-Fisher Equality). $\lambda_1 = \max_{\|x\|=1} x^T A x$.

3.5 Chebyshev’s Polynomial

Consider the following functions:

$$T_n(\cos \theta) = \cos (n \theta)$$

Exercise 3.21. Prove that $T_n$ is a polynomial of degree $n$. Find its leading coefficient.

These polynomials are called Chebyshev’s Polynomials of the first kind.

For example $T_0(x) = 0$.

$T_1(x) = x$.

$T_2(x) = 2x^2 - 1$. This follows from the fact $\cos(2\theta) = 2\cos^2(\theta) - 1$.

Exercise 3.22. 1. Calculate $T_3(x)$ and $T_4(x)$.

2. Give a recursive formula for $T_n$.

3. Show all roots of $T_n$ are real and are interlaced by the roots of $T_{n-1}$.

Exercise 3.23. Calculate the eigenvalues of the

1. Path of length $(n - 1)$.

2. Cycle of length $n$.

Exercise 3.24. Find the connection between the previous two exercises.