Apprentice Program: Linear Algebra

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We shall continue our discussion of eigenvectors and eigenvalues from last time.

Lemma 1. *Let A be a linear transformation. Then λ is an eigenvalue of A if and only if $ch(\lambda) = 0$. Here $ch(x)$ is the characteristic polynomial of A.*

Proof. λ is an eigenvalue $\leftrightarrow \exists v \neq 0$ such that $Av = \lambda v \leftrightarrow \exists v \neq 0$ such that $(\lambda I - A)v = 0 \leftrightarrow det(\lambda I - A) = 0 \leftrightarrow ch(\lambda) = 0$. □

Example 2. Consider $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then $det(\lambda I - A) = \lambda^2 - \lambda - 1$. This gives two eigenvalues $\lambda_1 = \frac{1+\sqrt{(5)}}{2}$ and $\lambda_1 = \frac{1-\sqrt{(5)}}{2}$. Now $v = (x, y)$ is an eigenvector for $\lambda_1 \leftrightarrow Av = \lambda v$. Since any eigenvector is nonzero multiple of another, we may assume $x = 1$. This give $y = \lambda_1$. Similarly $(1, \lambda_2)$ is an eigenvector for λ_2. We can write the matrix of eigenvectors $E = \left(\begin{smallmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{smallmatrix}\right)$. Then $AE = E\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$. Therefore $E^{-1}A^nE = (E^{-1}AE)^n$. This gives $A^n = E\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)E^{-1}$. This can be used in general to compute powers of A. Note that our particular choice of A applied to the vector $(1, 1)$ iteratively gives the Fibonacci sequence. This can be used to give a formula for the n-th term of the Fibonacci sequence.

Lemma 3. *Let $v_1, \ldots, v_k$ be eigenvectors of a linear transformation A with pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then the v_i are linearly independent.*

Proof. Suppose not. Let $\sum_1^k c_i v_i = 0$ be the relation with minimal number of nonzero constants c_i. Fix some nonzero c_j and apply $A - \lambda_j I$ to the above relation. This gives a relation of with a strictly smaller number of nonzero coefficients. The fact that it is a nontrivial relation follows from the fact that the eigenvalues are pairwise distinct so not all the coefficients can be zero. □

1 Jordan Form

It is not always possible to find a basis for a linear transformation consisting only of eigenvectors. The matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

as characteristic polynomial λ^2 with a single root, 0. So the only eigenvalue of this matrix is 0. Since the linear transformation corresponding to this matrix is not the linear transformation that takes the entire vector space to zero, the eigenspace corresponding to 0 cannot be the entire space. The eigenvector associated to 0 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

This shows that not every matrix is diagonalizable (i.e., can be written with respect to a basis of eigenvectors), but this is an example of the worst possible case.

Theorem 4 (Jordan Form). *Over an algebraically closed field, every linear transformation can be written as a matrix of the form*

$$\begin{pmatrix} B_1 & 0 & 0 & \dots & 0 \\ 0 & B_2 & 0 & \dots & 0 \\ 0 & 0 & B_3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & 0 & B_n \end{pmatrix}$$

where each B_i is an $n_i \times n_i$ matrix of the form

$$\begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & & \vdots \\ 0 & 0 & \lambda_i & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 & \lambda_i \end{pmatrix}$$

Note: The λ_i do not have to be distinct.

Given a linear transformation ϕ that can be written as a matrix of the form

$$\begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

with all other entires 0, the subspace corresponding to the top left block of all zeros is the kernel of ϕ. The subspace corresponding to the bottom right block is the kernel of $\phi - id$.

If ψ is a linear transformation that can be written as a matrix of the form

$$\begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ & & & 2 & 1 & 0 \\ & & & 0 & 2 & 1 \\ & & & 0 & 0 & 2 \end{pmatrix}$$

with all other entires zero, the subspace corresponding to the top block is the kernel of ψ^n for some large n. The subspace corresponding to the bottom block is the kernel of $(\psi - 2 \cdot id)^n$ for some large n.

Definition 5. If V is a vector space of dimension n and $\phi \in \mathrm{Hom}(V)$, $V_\lambda = \ker(\phi - \lambda \cdot id)^n$ is a *generalized eigenspace.*

Steps in the proof of Jordan Form. Let $\phi \in \mathrm{Hom}(V)$.

- Let $\lambda_1, \lambda_2 \ldots \lambda_k$ be the (distinct) roots of the characteristic polynomial of ϕ. If $v_1 \in V_{\lambda_1}, \ldots v_k \in V_{\lambda_k}$ and $v_i \neq 0$, then the v_i are linearly independent.

- $\mathrm{Span}(V_{\lambda_1}, \ldots, V_{\lambda_k}) = V$

  Corollary 6. $V = \oplus_{i=1}^k V_{\lambda_i}$

  By considering each generalized eigenspace separately, we can assume that ϕ has only one eigenvalue. We can further reduce to the case that the eigenvalue is zero by replacing ϕ by $\phi - id$. So we can assume that ϕ is nilpotent (i.e., $\phi^n = 0$ for some n).

Definition 7. For $v \in V$, a *chain* is $\{v, \phi(v), \phi^2(v), \dots, \phi^k(v)\}$ where $\phi^k(v) \neq 0$ and $\phi^{k+1}(v) = 0$.

- Each chain is ϕ-invariant (i.e., applying ϕ to an element of

  $$< v, \phi(v), \phi^2(v), \dots, \phi^k(v) >$$

  is an element of $< v, \phi(v), \phi^2(v), \dots, \phi^k(v) >$).

  Each chain is linearly independent. (If there were c_i, not all zero, so that $c_1 v_1 + \dots c_k \phi^k(v) = 0$ applying ϕ many times would lead to a contradiction.)

- For every chain of maximal length $\{v, \phi(v), \phi^2(v), \dots, \phi^k(v)\}$ let $W =< v, \phi(v), \phi^2(v), \dots, \phi^k(v) >$. Then there exists $U \subset V$ such that $U \cap W = 0$ and $\text{span}(U, W) = V$ (i.e. $V = U \oplus W$) and U is ϕ-invariant.

□

2 Change of Basis

Let V be a vector space and $E = \{e_1, \dots, e_n\}$ and $F = \{f_1, \dots f_n\}$ bases of V. Each $e_i = \sum_{j=1}^{n} c_{ji} f_j$ for some constants c_{ji}. These c_{ji} define a matrix B, called the *change of basis* matrix.

If $\phi \in \text{Hom}(V)$ and ϕ_E and ϕ_F are the matrices of ϕ corresponding to E and F then $\phi_E = B^{-1} \phi_F B$. This can be used to show that each linear transformation has a characteristic polynomial that does not depend on the matrix used to represent the linear transformation.