3b.1 Binomial Theorem

Definition 3b.1.1. \( \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \)

Throughout \( p \) will be a prime.

Exercise 3b.1.2. If \( p \) is prime, \( 1 \leq k \leq p-1 \) then \( p \mid \binom{p}{k} \).

Theorem 3b.1.3 (Binomial Theorem).

\[
(x + y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k}. \tag{3b.1.1}
\]

If you haven’t seen a proof of the Binomial Theorem then prove it as an exercise.

Exercise 3b.1.4. Let \( a \) and \( b \) be integers. Then \( (a + b)^p \equiv a^p + b^p \pmod{p} \).

Exercise 3b.1.5. Use the preceding exercise to prove Fermat’s little Theorem.

As a consequence of the binomial theorem one has the following identity:

\[
\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1 + 1)^n = 2^n. \tag{3b.1.2}
\]

We can also give a combinatorial proof of this identity. Let \( A \) be a set of size \( n \). Consider the subsets of size \( k \) of \( A \), here \( 0 \leq k \leq n \). This sum represents the left side of the above identity. On the other hand, giving a subset of \( A \) is equivalent to assigning 0 or 1 to each \( 1 \leq k \leq n \). The 0 or 1 tells us whether or not the given element is in the subset. There are \( 2^n \) such choices, giving the right half of the identity.
Let us say that a set is even if it has an even number of elements; and odd if it has an odd number of elements.

Let us now count the even subsets of $A$. One might guess that this number is half the total number of subsets. Indeed, it is $2^{n-1}$. We shall give two proofs of this fact.

Observe that the number of even subsets is $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}$.

**Theorem 3b.1.6.** For all $n > 0$, the number of even subsets of $A$ is equal to the number of odd subsets of $A$.

*Proof.* Note that the number of odd subsets is $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k+1}$. Applying the binomial theorem we get that the alternating sum of the binomial coefficients is $0: \binom{n}{0} - \binom{n}{1} + \cdots \pm \binom{n}{n} = (1-1)^n = 0^n = 0$. Note, here we use the fact that $n > 0$ as by our convention $0^0 = 1$. □

When $n$ is odd we can give an explicit bijection between the even subsets and the odd subsets. If $S$ is an even subset then its complement is odd and vice versa. So for odd $n$, complementation provides a bijection between even and odd subsets. The next exercise asks find a bijection that works for all $n > 0$.

**Exercise 3b.1.7.** If $A$ is a nonempty set then give a bijection between the even subsets and the odd subsets of $A$.

**Exercise* 3b.1.8.** Let $N(n, 3)$ denote number of subsets of $A$ whose size is divisible 3. Then $|N(n, 3) - \frac{2^n}{3}| < 1$.

### 3b.2 Chebyshev’s Theorem

Now we recall the Prime Number Theorem. Recall $\pi(x)$ denotes the number of primes less than or equal to $x$.

**Theorem 3b.2.1 (Prime Number Theorem).** $\pi(x) \sim \frac{x}{\ln(x)}$.

A weaker version of this theorem was proved by Chebyshev:

**Theorem 3b.2.2 (Chebyshev’s Theorem).** There exist positive constants $c_1$ and $c_2$ such that $c_1 \frac{x}{\ln(x)} < \pi(x) < c_2 \frac{x}{\ln(x)}$ for all $x \geq 2$.

While no simple proof of the Prime Number Theorem is known, we shall prove Chebyshev’s theorem in the next class. We list a series of exercises geared towards this proof.

**Exercise 3b.2.3.** $4^n < \frac{(2^n)^n}{2n+1} < 4^n$.

**Exercise 3b.2.4.** $\binom{2n+1}{n} < 4^n$.

**Exercise 3b.2.5.** Find the exponent of the prime $p$ in $n!$ (i.e., find the largest $k$ such that $p^k | n!$).

**Exercise 3b.2.6.** Prove: if $p^\ell$ divides $\binom{n}{k}$ then $p^\ell \leq n$.  

---

2
Exercise* 3b.2.7. Show that $\prod_{p \leq x} p \leq 4^x$.

Hint: Observe that

$$\prod_{k+2 \leq p \leq 2k+1} p \mid \binom{2k+1}{k}.$$  \hspace{1cm} (3b.2.1)

Exercise 3b.2.8. Use Exercise 3b.2.7 to prove the upper bound portion of Chebyshev’s theorem: there exists $C > 0$ such that $\pi(x) < C \frac{x}{\ln(x)}$.

Finally, a couple of unrelated exercises.

Exercise 3b.2.9. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Exercise 3b.2.10. (Experimental exercise.) Draw a large chunk of the Pascal triangle mod 2. Observe the pattern and make conjectures. Prove some of your conjectures.