REU 2006 Apprentice Problem Sheet 2

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due Wednesday, July 5, 2006

This problem set begins with a long list of linear algebra problems and concludes with some problems in number theory.

1. Are there infinitely many vectors in $\mathbb{R}^3$ such that any three are linearly independent?

2. Is the set $\{1, \sqrt{2}, \sqrt{3}\}$ linearly independent over $\mathbb{Q}$?

3. Let $\alpha_1, \ldots, \alpha_n$ be distinct real numbers. Prove that the set

$$\left\{ \frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2}, \ldots, \frac{1}{x-\alpha_n} \right\}$$

   of rational functions is linearly independent over $\mathbb{R}$.

4. Let $A_1 A_2 \ldots A_n$ be a regular n-gon inscribed in the unit circle. What is the product of the distances from A_1 to the other vertices?

5. Let V be a vector space over the field K. True or false?
   a) Every subset of a linearly independent set is linearly independent;
   b) Every nonempty set of vectors has a nonempty linearly independent subset;
   c) Every independent subset is contained in a basis;
   d) The union of two subspaces U, W is a subspace if and only if $U \subseteq W$ or $W \subseteq U$.

6. The dragon appears to the princess at midnight, gives her a 13×21 real matrix of rank 8 and says: "Every morning you can change an entry of my matrix. I will come every midnight and can also change an entry. If the rank ever goes down to 7, I shall eat you." Would it help the princess to take a quick course in linear algebra?

7. Let A be an $n \times n$ integer valued matrix such that the sum of every row is divisible by 7. Show that $\det(A)$ is divisible by 7.

8. Let A be an $n \times n$ matrix. Show that if there exists an $m \times k$ submatrix which is all 0 and $m + k > n$ then $\det(A) = 0$.

9. **(Hilbert matrix)** Let $a_1, a_2, \ldots, a_n$ be a list of n distinct numbers and $b_1, b_2, \ldots, b_n$ another list of n distinct numbers. Consider the $n \times n$ matrix $H = (h_{ij})$ with
$$h_{ij} = \frac{1}{a_i + b_j}.$$
Prove that the rows of H are linearly independent.

10. A permutation $\pi \in \mathrm{Sym}(X)$ is fixed-point-free if for all $x \in X$ we have $x^\pi \neq x$. Are there more fixed-point-free even permutations on 100 points than odd ones?

11. Show that the equation $AB - BA = I$ is unsolvable among the $n \times n$ complex matrices. (I is the identity matrix.)

12. Find an $n \times n$ matrix M such that $M^n = 0$ but $M^{n-1} \neq 0$.

13. What happens to the determinant if we reflect the matrix in its anti-diagonal?

14. Show that if A is a 2×2 matrix then
$$A^2 - \mathrm{tr}(A)A + \det(A)I = 0.$$
Here $\mathrm{tr}(A)$ is the trace of A, that is, the sum of the diagonal entries in A.

15. Show that for every M there exists a polynomial $p(x)$ such that $p(M) = 0$, in the above sense.

16. **(Vandermonde determinant)** Let $a_1, a_2, \ldots, a_n$ be numbers. The Vandermonde matrix with generators $a_1, a_2, \ldots, a_n$ is the $n \times n$ matrix
$$V(a_1, a_2, \ldots, a_n) = \begin{bmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \ldots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{bmatrix}.$$
Prove:
$$\det(V(a_1, a_2, \ldots, a_n)) = \prod_{1 \leq i < j \leq n} (a_i - a_j).$$

17. Let a, b be numbers. Verify this determinant evaluation:
$$\det \begin{bmatrix} a & b & b & \ldots & b \\ b & a & b & \ldots & b \\ b & b & a & \ldots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \ldots & a \end{bmatrix} = (a-b)^{n-1}(a + b(n-1))$$

18. Is it true that if $AB = 0$ then $BA = 0$? (A, B are $n \times n$ matrices.)

19. What is the sum of the primitive n-th roots of unity?

20. Let $a_{ij} = \gcd(i, j)$ $(1 \le i, j \le n)$. Prove that for the $n \times n$ matrix $A = (a_{ij})$ we have

$$\det(A) = \varphi(1) \cdot \varphi(2) \cdot \dots \cdot \varphi(n)$$

21. Let r be the probability that two random positive integers are relatively prime. Recall that this value is defined as a limit. Assuming the limit exists, i. e., assuming that r is well defined, give an AH-HA proof that $r = 6/\pi^2$.

22. Prove that there are infinitely many primes of the form $4k - 1$.

23. Prove that $\sum' 1/n$ is finite, where the summation is extended over all integers which do not have the string 2006 in their decimal representation.

24. (a) Prove that there are infinitely many primes which begin with the digits 2006 (in decimal). (b) Prove that the sum of the reciprocals of these primes diverges.

25. A polynomial $f(x)$ is "integer-preserving" if $f(x)$ is an integer whenever x is an integer. An integer-preserving polynomial is "congruence preserving" if $f(a) \equiv f(b) \pmod m$ whenever $a \equiv b \pmod m$, for all triples of integers a, b, m. An *integral polynomial* is a polynomial with integer coefficients. Note that every integral polynomial is integer-preserving.

    (a) Find an integer-preserving polynomial which is not integral.

    (b) Prove that every integral polynomial is congruence preserving.

    (c) Find a congruence preserving polynomial which is not integral.