REU 2006 Apprentice Problem Sheet 3

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NEW PROBLEMS

1. (a) Prove that every prime other than 2 and 3 is $\equiv \pm 1 \pmod 6$.

   (b) Prove that there are infinitely many primes $\equiv -1 \pmod 6$.

   (c) Prove: if x is an integer, p is a prime, and $p \mid x^2 + x + 1$ then either $p = 3$ or $p \equiv 1 \pmod 6$.

   (d) Prove that there are infinitely many primes $\equiv 1 \pmod 6$.

2. Let $\sigma(n)$ denote the sum of the positive divisors of n. (For instance, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.) Find a value of n such that $\sigma(n) > 100n$.

3. Let
$$N(n, 3) = \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n}{3k}.$$
Prove:
$$\left| N(n, 3) - \frac{2^n}{3} \right| < 1. \tag{1}$$

4. Use the Prime Number Theorem to solve the problems (a), (b), (c), (e) below.

   (a) $p_n \sim n \ln n$, where p_n denotes the n-th prime.

   (b) Show that the average of all primes $\le x$ is asymptotically $x/2$:
$$\frac{\sum_{p \le x} p}{\pi(x)} \sim \frac{x}{2}. \tag{2}$$
   (The summation is over all primes $p \le x$.)

   (c) Let $x(n)$ denote the largest integer value of x such that $\sum_{p \le x} p \le n$. Prove: $x(n) \sim \sqrt{(n \ln n)/2}$.

   (d) Asymptotically evaluate (i) the arithmetic mean and (ii) the geometric mean of the numbers $1, 2, \ldots, x$.

(e) Can you evaluate asymptotically the geometric mean of all primes $\le x$? The quantity in question is $(\prod_{p\le x} p)^{1/\pi(x)}$. (Note that the geometric mean is less than the arithmetic mean, which is $\sim x/2$ according to part (b).)

5. Prove that for every function $f : \mathbb{F}_p \to \mathbb{F}_p$ there exists a polynomial $p(x) \in \mathbb{F}_p[x]$ such that $f(x) = p(x)$ for all $x \in \mathbb{F}_p$.

6. Show that if $M^k = 0$ for some k (we call these matrices nilpotent) then $I - M$ is invertible, that is, there exists A such that $(I - M)A = I$.

7. How many invertible $n \times n$ matrices are there over $\mathbb{F}_p$?

8. What are the eigenvalues, eigenvectors and diagonal form of the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

over the complex numbers?

9. Let $a_0, a_1, \dots, a_{n-1}$ be complex numbers, let $f(x) = a_0 + a_1 x + \dots + a_{n-1}x^{n-1}$ and let $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$ be the n-th complex roots of unity. Show that the determinant

$$\det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix} = f(\varepsilon_0) \cdot f(\varepsilon_1) \cdot \dots \cdot f(\varepsilon_{n-1})$$

10. Find a vectorspace V and linear transformations $A, B : V \to V$ such that $AB = \mathrm{Id}_V$ but $BA \neq \mathrm{Id}_V$.

LEFTOVERS from Problem sheet 1

11. There are 17 weights with the property that if you omit any of them you can divide the rest into two equal sized groups, such that the sum weights of the two groups are equal. Show that all the weights are equal. *Hint.* This is a linear algebra problem. (a) Prove the statement when all weights are integers. (b) Reduce the general case (real weights) to the case of integer weights.

12. A necklace is an arrangement of n beads around a circle. We have two kinds of beads, red and blue. Other than their color, the beads are identical. Two necklaces do not count as distinct if one is obtained from the other by rotation.

    Determine the number of necklaces made of n beads where

    (b) $n = p^2$ where p is a prime number;

    (c) $n = pq$ where p, q are distinct primes;

    (d) $n = pqr$ where p, q, r are distinct primes.

    Generalize your answers to the case when the beads come in k colors. In each case, your answer should be a simple closed-form expression (no summation symbols or dot-dot-dots).

13. Can you cover a 100×100 board with 8×1 "dominoes"?

14. You are given a pair of integers (a, b). A step is to add an integer multiple of one of the entries to the other entry. Can you always reach $(0, *)$ in at most 100 steps?

15. Show that every sequence of $n^2 + 1$ distinct real numbers contains an increasing or a decreasing subsequence of length $n + 1$.

16. Assume that a polynomial f maps rationals to rationals. Show that f has rational coefficients.

17. Are there two infinite subsets A and B of the nonnegative integers such that every nonnegative integer can be uniquely written as the sum of an element of A and an element of B?

18. Consider the 8×8 chessboard. Some of the 64 cells are infected. If a cell has at least 2 infected neighbours it becomes infected. (Two cells are neighbors if they share a side.) An infected cell is never cured. Show that you cannot infect the full board with fewer than 8 initially infected cells. (Note. This is an AH-HA problem. The essence of the solution is contained in a single word. Discover that word.)

19. Show that for every natural number n the equation

    $$\sum_{i=1}^{n} \frac{1}{a_i} = 1$$

    has only a finite number of solutions in natural numbers a_i.

LEFTOVERS from Problem sheet 2

20. Is the set $\{1, \sqrt{2}, \sqrt{3}\}$ linearly independent over $\mathbb{Q}$?

21. Let $\alpha_1, \dots, \alpha_n$ be distinct real numbers. Prove that the set

$$\left\{\frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2}, \dots, \frac{1}{x-\alpha_n}\right\}$$

of rational functions is linearly independent over $\mathbb{R}$.

22. The dragon appears to the princess at midnight, gives her a 13×21 real matrix of rank 8 and says: "Every morning you can change an entry of my matrix. I will come every midnight and can also change an entry. If the rank ever goes down to 7, I shall eat you." Would it help the princess to take a quick course in linear algebra?

23. Let A be an $n \times n$ matrix. Show that if there exists an $m \times k$ submatrix which is all 0 and $m + k > n$ then $\det(A) = 0$.

24. **(Hilbert matrix)** Let $a_1, a_2, \dots, a_n$ be a list of n distinct numbers and $b_1, b_2, \dots, b_n$ another list of n distinct numbers. Consider the $n \times n$ matrix $H = (h_{ij})$ with

$$h_{ij} = \frac{1}{a_i + b_j}.$$

Prove that the rows of H are linearly independent.

25. A permutation $\pi \in \mathrm{Sym}(X)$ is fixed-point-free if for all $x \in X$ we have $x^\pi \neq x$. Are there more fixed-point-free even permutations on 100 points than odd ones? (Hint: this is a determinant problem.)

26. Show that the equation $AB - BA = I$ is unsolvable among the $n \times n$ complex matrices. (I is the identity matrix.)

27. Find an $n \times n$ matrix M such that $M^n = 0$ but $M^{n-1} \neq 0$.

28. What happens to the determinant if we reflect the matrix in its anti-diagonal?

29. Show that for every M there exists a polynomial $p(x)$ such that $p(M) = 0$, in the above sense.

30. **(Vandermonde determinant)** Let $a_1, a_2, \dots, a_n$ be numbers. The Vandermonde matrix with generators $a_1, a_2, \dots, a_n$ is the $n \times n$ matrix

$$V(a_1, a_2, \dots, a_n) = \begin{bmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \dots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{bmatrix}.$$

Prove:

$$\det(V(a_1, a_2, \dots, a_n)) = \prod_{1 \le i < j \le n} (a_i - a_j).$$

31. Let a, b be numbers. Verify this determinant evaluation:

$$\det \begin{bmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix} = (a-b)^{n-1}(a+b(n-1))$$

32. Prove: the sum of the primitive n-th roots of unity is $\mu(n)$ where μ denotes the Möbius function.

33. Let $a_{ij} = \gcd(i, j)$ $(1 \le i, j \le n)$. Prove that for the $n \times n$ matrix $A = (a_{ij})$ we have
$$\det(A) = \varphi(1) \cdot \varphi(2) \cdot \dots \cdot \varphi(n).$$

34. Let r be the probability that two random positive integers are relatively prime. Recall that this value is defined as a limit. Assuming the limit exists, i. e., assuming that r is well defined, give an AH-HA proof that $r = 6/\pi^2$.

35. Prove that $\sum' 1/n$ is finite, where the summation is extended over all integers which do not have the string 2006 in their decimal representation.

36. (a) Prove that there are infinitely many primes which begin with the digits 2006 (in decimal).
    (b) Prove that the sum of the reciprocals of these primes diverges.

37. A polynomial $f(x)$ is "integer-preserving" if $f(x)$ is an integer whenever x is an integer. An integer-preserving polynomial is "congruence preserving" if $f(a) \equiv f(b) \pmod m$ whenever $a \equiv b \pmod m$, for all triples of integers a, b, m. An *integral polynomial* is a polynomial with integer coefficients. Note that every integral polynomial is integer-preserving.
    (a) Find an integer-preserving polynomial which is not integral.
    (b) Prove that every integral polynomial is congruence preserving.
    (c) Find a congruence preserving polynomial which is not integral.