A6.1 Convexity

Definition A6.1.1. The line segment in $\mathbb{R}^n$ connecting two vectors $x$ any $y$ is defined to be $\{ax + by \mid a, b \geq 0, a + b = 1\}$.

Definition A6.1.2. A set $S \subseteq \mathbb{R}^n$ is said to be convex if for every $x, y \in S$, the line segment connecting $x$ and $y$ is also in $S$.

Definition A6.1.3. Let $S \subseteq \mathbb{R}^n$. The convex hull of $S$, denoted $\text{conv} S$, is the smallest convex set containing $S$. That is, if a convex set $T$ is such that $S \subseteq T$, then $\text{conv} S \subseteq T$.

Remark A6.1.4. Given a set $S \subseteq \mathbb{R}^n$, $\text{conv} S$ exists.

Exercise A6.1.5. Let $I$ be an arbitrary (possibly infinite) indexing set. Let $\{C_i\}_{i \in I}$ be a collection of convex sets in $\mathbb{R}^n$. Then $\cap_{i \in I} C_i$ is again a convex set.

Definition A6.1.6. Let $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$. A convex combination of $\{v_1, \ldots, v_k\}$ is a linear combination, 

$$\sum_{i=1}^{k} \alpha_i v_i,$$

such that $\alpha_i \geq 0$ for $i = 1, \ldots, k$, and $\sum_{i=1}^{k} \alpha_i = 1$.

Exercise A6.1.7. Show that $\text{conv} S = \{\text{convex combinations of elements in } S\}$. 
A6.2 Affine subspaces

**Definition A6.2.1.** An affine combination of \( v_1, \ldots, v_k \) is a linear combination,

\[
\sum_{i=1}^{k} \alpha_i v_i,
\]

such that \( \sum \alpha_i = 1 \).

**Remark A6.2.2.** Of course, the above definition makes sense for a vector space \( V \) over any field \( F \).

**Remark A6.2.3.** The difference between a convex combination and an affine combination is that the coefficients in the latter can be negative. Thus, in by affine combinations we get the entire “flat” (line, plane, etc.) containing \( \{v_1, \ldots, v_k\} \).

**Definition A6.2.4.** Let \( U \subseteq V \) be a subset of a vector space \( V \). We say that \( U \) is an affine subspace if \( U \) is closed under affine combinations. This includes the possibility that \( U \) is empty.

To distinguish from “affine subspaces,” we shall refer to the subspaces of a vector space as “linear subspaces.” So a linear subspace necessarily includes the origin.

**Remark A6.2.5.** Every affine subspace can be associated with a unique linear subspace, namely the subspace parallel to the given affine hyperplane. We shall make this precise below.

**Exercise A6.2.6.** The intersection of (possibly infinitely many) affine subspaces is an affine subspace.

**Definition A6.2.7.** The affine closure of \( S \subseteq V \), denoted \( \text{aff } S \), is the smallest affine subspace containing \( S \).

Analogous to the case of convex sets, by Exercise A6.2.6 affine closures exist.

**Exercise A6.2.8.** Given \( S \subseteq V \), show that

\[
\text{aff } S = \{ \text{all affine combinations of } S \}.
\]

**Remark A6.2.9.** In the case where \( S = \emptyset \), \( \text{aff } \emptyset = \emptyset \). But, \( \text{Span } \emptyset = \{0\} \).

**Exercise A6.2.10.** If \( U \) is a nonempty affine subspace of a vector space \( V \), then there exists a unique linear subspace \( W \subseteq V \) and there exists \( u \in V \) such that \( U = W + u \). (Hint: Any \( u \in U \) will do the job.)

**Remark A6.2.11.** Recall from group theory that \( W + u \) is a coset of \( W \), that is, an element of \( W \setminus V \). In fact, if \( U = W + u \) and \( U = W + u' \), then \( u - u' \in W \). Thus, there is a well-defined map from the set of affine subspaces parallel to \( W \) to the quotient space \( V/W \).

To show that these two sets are in fact in bijection, we need:

**Exercise A6.2.12.** Conversely, all cosets of a linear subspace are affine subspaces.
A6.3 Helly’s Theorem and Helly-type theorems

Exercise A6.3.1 (Helly’s Theorem for \( \mathbb{R}^2 \)). If \( C_0 \ldots C_k \) are convex sets in \( \mathbb{R}^2 \), \( k \geq 2 \), with the property that every 3 of the \( C_i \) share a point, then

\[
\bigcap_{i=1}^{k} C_i \neq \emptyset.
\]

Remark A6.3.2. If we suppose that only every 2 intersect, then the above is no longer true. For example, consider the sides of a triangle.

We prove the following simpler case of Helly’s Theorem.

Proposition A6.3.3. The analog of Helly’s Theorem is true for \( \mathbb{R}^1 \). That is, given \( C_0 \ldots C_k \), convex sets in \( \mathbb{R}^1 \), \( k \geq 1 \), with the property that every 2 of the \( C_i \) share a point, then

\[
\bigcap_{i=1}^{k} C_i \neq \emptyset.
\]

Proof. First we prove the result for \( k = 3 \).

Let \( P_1 \in C_2 \cap C_3 \), \( P_2 \in C_1 \cap C_3 \), and \( P_3 \in C_1 \cap C_2 \). Assume, without loss of generality, that \( P_1 \leq P_2 \leq P_3 \). Then, since \( P_1, P_3 \in C_2 \), and \( C_2 \) is convex, the line segment joining \( P_1 \) and \( P_2 \) is also in \( C_2 \). Thus, \( P_2 \in C_2 \) and therefore \( P_2 \in C_1 \cap C_2 \cap C_3 \).

Now we prove the result by induction on \( k \).

There is nothing to prove for \( k \leq 2 \), and we just proved the result for \( k = 3 \). Assume now that \( k \geq 4 \) and that the result holds for \( k - 1 \) intervals.

Replace \( C_1 \ldots C_k \) by \( D_1, \ldots, D_{k-1} \), where \( D_i = C_k \cap C_i \neq \emptyset \). Now \( D_i \cap D_j = C_i \cap C_j \cap C_k \neq \emptyset \) by the result for 3 terms. Therefore, by induction,

\[
\bigcap_{i=1}^{k} D_i \neq \emptyset.
\]

But,

\[
\bigcap_{i=1}^{k-1} D_i = \bigcap_{i=1}^{k} C_i.
\]

Exercise A6.3.4. Every pair of longest paths in a connected graph shares a vertex.

Exercise A6.3.5. The set of all longest paths in a tree share a vertex.

Exercise* A6.3.6. Find a connected graph in which the set of all longest paths does not share a vertex.
Exercise A6.3.7. In a tree, if $F_1 \ldots F_k$ are subtrees and for every $i,j$, $F_i$ and $F_j$ share a vertex, then all the $F_i$ share a vertex. This is a generalization of the 1-dimensional version of Helly’s Theorem.

Remark A6.3.8. Let $T$ be a tree. Then the convex subsets, $F \subseteq T$ are exactly the subtrees. Here $F$ convex means that for every pair of vertices $v_1, v_2 \in F$ the unique path connecting $v_1$ and $v_2$ is also in $F$.

A6.4 A vector representation of graphs

Exercise A6.4.1. Let $G$ be a graph with vertex set $\{1, \ldots, n\}$. Let $v_1, \ldots, v_n$ be linearly independent vectors in a vector space $V$ (over any field). With an edge $\{i,j\}$ let us associate the vector $v_i - v_j$. What property of the graph characterizes when the vectors associated with the edges are linearly independent? (The answer is a very simple graph property.)

A6.5 Linear Independence

In this section we will always denote by $V$ a vector space over a field $F$.

Theorem A6.5.1 (Miracle 1). If $v_1, \ldots, v_k$ are linearly independent vectors in $V$, and $w_1, \ldots, w_m$ are generators of $V$, then $k \leq m$.

To show this, we need:

Lemma A6.5.2 (Steinitz Exchange Principle). If $v_1, \ldots, v_k$ are linearly independent and $w_1, \ldots, w_n$ span $V$, then there exists some $j$ such that $v_1, \ldots, v_{k-1}, w_j$ are linearly independent.

We review some definitions.

Definition A6.5.3. We say $v_1, \ldots, v_k$ are linearly independent if only their trivial linear combination is zero. In other words, $(\forall \alpha_1, \ldots, \alpha_k \in F)(\text{if } \sum_{i=1}^{k} \alpha_i v_i = 0 \text{ then } \alpha_1 = \cdots = \alpha_n = 0.)$

Definition A6.5.4. We say $v_1, \ldots, v_k$ are called linearly dependent if they are not linearly independent.

Definition A6.5.5. We say $w$ depends on $\{v_1, \ldots, v_k\}$ if there are $\alpha_1, \ldots, \alpha_k \in F$ such that

$$\sum_{i=1}^{k} \alpha_i v_i = w.$$ 

Observation A6.5.6. $v_1, \ldots, v_k$ are linearly dependent if and only if there exists $j$ such that $v_j$ depends on $\{v_i : i \neq j\}$.

Definition A6.5.7. If $S, T \subseteq V$, we say that $S$ depends on $T$ if every element of $S$ depends on $T$. Equivalently, if and only if $S \subseteq \text{Span } T$. 

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Lemma A6.5.8. We have $S \subseteq \text{Span } T$ if and only if $\text{Span } S \subseteq \text{Span } T$.

Proof. (Scribe’s version) To paraphrase Gertrude Stein: The span of a span is a span. (The original quote is "A rose is a rose is a rose.")

Corollary A6.5.9. If $S_1$ depends on $S_2$ and $S_2$ depends on $S_3$ then $S_1$ depends on $S_3$.

Lemma A6.5.10. If $v_1, \ldots, v_k$ are linearly independent, and $v_1, \ldots, v_k, v_{k+1}$ are linearly dependent, then $v_{k+1}$ depends on $v_1, \ldots, v_k$.

Proof. By definition there exist $\alpha_1, \ldots, \alpha_{k+1} \in F$ such that
\[ \sum_{i=1}^{k+1} \alpha_i v_i = 0. \]
Then $\alpha_{k+1}$ cannot be 0 because $v_1, \ldots, v_k$ are linearly independent. Thus, we can divide through by $-\alpha_{k+1}$ and add $v_{k+1}$ to both sides to express $v_{k+1}$ as a linear combination of $v_1, \ldots, v_k$.

Proof of Steinitz’s Exchange Principle. We prove by contradiction. Suppose that for all $j$, $v_1, \ldots, v_{k-1}, w_j$ are linearly dependent. Then by Lemma A6.5.10, the vector $w_j$ is linearly dependent on $\{v_1, \ldots, v_{k-1}\}$. This holds for each $j$, thus $\text{Span } v_1, \ldots, v_{k-1} = V$. Thus, $v_k$ depends on $\{v_1, \ldots, v_{k-1}\}$, a contradiction.

Observation A6.5.11. If $v_1, \ldots, v_k$ are linearly independent, then for every $i \neq j$, $v_i \neq v_j$.

Proof of Miracle 1. Inductively, we can replace $v_i$ with $w_{ji}$, $i = 1, \ldots, k$, as in the Exchange Principle. We then get a list $w_{j1}, \ldots, w_{jk}$ of $k$ linearly independent vectors from among the $w_j$. By above observation, these must all be distinct.

A6.6 Left and Right Inverses

Definition A6.6.1. A semigroup is a set $S$ with an associative binary operation denoted by juxtaposition satisfying:

- For every $a, b \in S$, there exists a unique $ab \in S$.
- For every $a, b, c \in S$, $(ab)c = a(bc)$.

Definition A6.6.2. A monoid is a semigroup with an identity. That is, there exists $e \in S$ such that for any $a \in S$, $ae = ea = a$.

Definition A6.6.3. Let $S$ be a monoid. We say that $y \in S$ is a left (resp., right) inverse of $x \in S$ if $yx = e$ (resp., $xy = e$).

Exercise A6.6.4. (This problem was erroneously stated in class.)

(a) Prove: in a finite monoid, if an element has a left inverse then it has a right inverse.

(b) Find a monoid with an element that has a left inverse but not a right inverse.
Lemma A6.6.5. If $S$ is a monoid, and $x \in M$ has both a left inverse, $y$, and a right inverse, $z$, then $y = z$.

Proof. Consider the element $yxz$.

Our next goal is to show that:

Theorem A6.6.6. If an $n \times n$ matrix has a right inverse, then it has a left inverse (and so the two are equal by Lemma A6.6.5).

Suppose $A$ and $B$ are two $n \times n$ matrices, and let $I$ denote the identity $n \times n$ matrix. Write $A = \begin{bmatrix} a_1, \ldots, a_n \end{bmatrix}$, $B = \begin{bmatrix} b_1, \ldots, b_n \end{bmatrix}$, and $I = \begin{bmatrix} e_1, \ldots, e_n \end{bmatrix}$ (listing the columns of each matrix).

Suppose that $AB = I$. Then $Ab_i = e_i$ for each $i = 1, \ldots, n$. Thus, $e_i \in \text{Span} a_1, \ldots, a_n$, so, $\text{Span} a_1, \ldots, a_n = F^{m\times n}$. Since $\dim F^n = n$, we must have that $a_1, \ldots, a_n$ are linearly independent. That is, $\text{rk} \text{col} A = n$.

Conversely, if $\text{rk} \text{col} A = n$ then the columns of $A$ are linearly independent and therefore each $e_i$ depends on them, i.e., for each $i$ there exists $b_i$ such that $Ab_i = e_i$; putting these $b_i$ together we find that $A$ has a right inverse $B$.

We conclude that the existence of a right inverse for $A$ is equivalent to $\text{rk} \text{col} A = n$.

By applying the result to $A^T$ we see that the existence of a left inverse is equivalent to $\text{rk} \text{row} A = n$.

Theorem A6.6.7 (Miracle 2). For any matrix $A$, $\text{rk} \text{col} A = \text{rk} \text{row} A$.

This concludes the proof of Theorem A6.6.6.