REU 2007 - Discrete Mathematics

Math Puzzles - First Set. June 18, 2007

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1. **(Guess your hat)** An athletic team has n members; their shirts are numbered 1 to n. Each member receives a hat; each hat is either red or blue. The hats are distributed at random, with a coin flip for each person. Nobody knows the color of their own hat but they can see everybody else's hat and they also see everybody's number (printed on the shirt) (including their own). Team members must guess the color of their hat or pass under the following rules. Each team member must simultaneously and independently check "red" or "blue" or "pass" on an answer sheet (no communication permitted). If everybody passes, the team loses. If at least one person guesses the wrong color, the team loses. If at least one person does not pass and each person who does not pass guesses the right color, the team wins a big prize. Before the hats are distributed, the team gets to have a long strategy session; their goal is to devise a strategy which maximizes the team's chance of winning.

   It should be clear that each person's guess is wrong 50% of the time, regardless of strategy. So what's the use of strategy? Can the team beat the odds?

   Surprisingly, yes. In fact, the winning chance approaches 1 as the team grows. Prove: for $n = 2^k - 1$, it is possible to achieve a winning chance of $1 - 1/(n + 1)$. (Prove this for $n = 3$ first; the probability of winning should then be 3/4.) (Hint: pairwise independence, perp.)

2. **(Cleaning the corner - Tom Hayes)** We label the cells of the positive quadrant (the "game board") by pairs of integers $\{(i, j) : i, j \geq 0\}$. The neighbor to the North of cell (i, j) is cell $(i + 1, j)$; the neighbor to the East is cell $(i, j + 1)$. The corner cell is $(0, 0)$. The Manhattan distance between cells (i_1, j_1) and (i_2, j_2) is $|i_1 - i_2| + |j_1 - j_2|$.

   Chips are placed on some of the cells, at most one chip per cell. Chips "shift and multiply" in the following manner: suppose a chip is on cell (i, j). If both its neighbor to the North and its neighbor to the East are empty, we can remove the chip from (i, j) and place a chip on its neighbor to the North and another chip on the neighbor to the East.

   Initially we put a chip on cell $(0, 0)$; otherwise the game board is empty. We wish to clean the corner, i. e., we wish to achieve, by a sequence of "shift/multiply" moves, that there be no chip left within Manhattan distance d from the corner. Prove that this is impossible (a) for $d = 3$; (b) for $d = 2$.

3. **(Spreading Infection)** Some of the 64 cells of a chessboard are initially infected. Subsequently the infection spreads according to the following rule: if two neighbors of a cell are infected then the cell gets infected. (Neighbors share and edge, so each cell has at most four neighbors.) No cell is ever cured. What is the minimum number of cells that need to be initially infected to guarantee that the infection spreads all over the chessboard? It is easy to see that 8 are sufficient in many ways. Prove that 7 are not enough. (This is an AH-HA problem.)

4. **(Dominoes)** Prove: if we remove two opposite corners from the chessboard, the board cannot be covered by dominoes. (Each dominoe covers two neighboring cells of the chessboard.) Look for an "Ah-ha" proof: clear, convincing, no cases to distinguish.

5. **(Triominoes)** Remove a corner from a 101×101 chessboard. Prove that the rest cannot be covered by *triominoes*. A triomino is like a domino except it consists of three squares in a row; each cell can cover one cell on a chessboard. Each triomino can either "stand" or "lie." Find an "Ah-ha" proof.

6. **(Mouse and cheese)** A mouse finds a $3 \times 3 \times 3$ chunk of cheese, cut into 27 blocks (cubes), and wishes to eat one block per day, always moving from a block to an adjacent block (a block that touches the previous block along a face). Moreover, the mouse wants to leave the center cube last. Prove that this is impossible. Find two "Ah-ha!" proofs; one along the lines of the solution of knight's trail problem, the other inspired by the solution of the dominoes problem.

7. **(Chocolate bar)** 96 kids wait for us to split an 8×12 chocolate bar along the grooves into 96 small rectangles. It is up to us in what order we do the splitting; we can start, for instance, by breaking the 7 long grooves and then split each of the 8 long (1×12) pieces; or we can start with the short grooves, or halve the bar each time, or any other way. The one thing we are not permitted to do is stack the pieces. At any one time, we have to pick up one piece and break it into two.

   Each break takes us 1 second. Find the fastest method. (This is another "Ah-ha" problem.)

8. **(Balancing numbers)** Suppose we have 13 real numbers with the following property: if we remove any one of the numbers, the remaining 12 can be split into two sets of 6 numbers each with equal sum. Prove: all the 13 numbers are equal. (Hint: first assume all the numbers are integers.)