REU 2007 - Discrete Mathematics

Math Puzzles - Second Set. June 20, 2007

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1. **(Dividing a rectange)** A large rectangle is cut up into a finite number of smaller rectangles. (All edges are either horizontal or vertical.) Suppose each of the smaller rectangles has at least one side of integer length. Prove that the same holds for the large rectangle.

2. **(Monotone subsequences: Erdős – Szekeres)** Prove: a sequence of $k\ell+1$ distinct real numbers necessarily contains either an increasing sequence of $k+1$ or a decreasing sequence of $\ell+1$ terms. (AH-HA)

3. **(Hilbert matrix)** Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be $2n$ distinct elements of a field F. Prove that the $n \times n$ matrix $H = (h_{ij})$ is nonsingular, where
$$h_{ij} = \frac{1}{a_i - b_j}.$$

4. **(Unit distances: Erdős)** Let us consider a set of n points in the plane; assume the distance between each pair is at most 1 unit. Prove: the number of pairs at unit distance is $\leq n$.

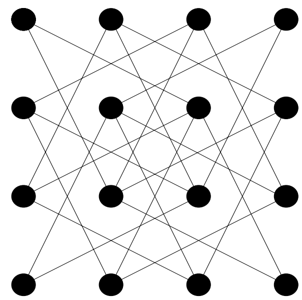
5. **(Coloring the unit distance graph)** Consider the "unit distance graph" in the plane. The vertices of this graph are all the points in the plane; two points are adjacent if they are at unit distance. Prove that the chromatic number of this graph is (a) at least four; and (b) at most 7. (Nothing more is known about the chromatic number of this graph.)

6. **(Intersecting lines: Sylvester)** Let us consider n lines in the plane, not all of which pass through a point. Prove: there is a point in which exactly two of the lines intersect. (AH-HA solution found by Gallai about 70 years after the problem was posed.)

7. **(Erdős test 1)** Let S be a set of $n+1$ integers from $\{1, 2, \ldots, 2n\}$. Prove that two of them are relatively prime. First show that this can be avoided if S has only n numbers. ("Ah-ha" again.)

8. **(Erdős test 2)** Let S be a set of $n+1$ integers from $\{1, 2, \ldots, 2n\}$. Prove that one of them divides another. First show that this can be avoided if S has only n numbers. (Yes, "Ah-ha" again. Which is not to say that the proof is easy to find. But it is very easy to understand.) **(Paul Erdős (1913-1996)** liked to ask these two questions of the *epsilons,* promising talents - high schoolers and even younger, introduced to him.)

9. **(Knight's trail)** Consider a knight moving around on a 4×4 chessboard. We let the knight start at any cell of our choosing, and we wish to guide it through 15 moves so it never steps on a previously visited cell. So, after the 15 moves, the knight will have visited each cell. Prove that this is impossible. Find an "Ah-ha" proof.



Graph of knight moves on a 4×4 chessboard.

10. **(Pennies on the table)** Alice and Bob alternate putting pennies on a round table (one penny per move). The pennies cannot overlap. The one who cannot place a penny loses. (There is an unlimited supply of pennies.) Prove: Alice has a winning strategy.

11. **(Divisor game)** We are given an integer $n \geq 2$. Alice and Bob alternate naming positive divisors of n (one divisor per move). No divisor of a previously named number can be named again. The one who says "n" loses. Prove: Alice (the first player) has a winning strategy.

12. **(Picking coins: Who has the advantage?)** A row of 100 coins is placed on a table. Alice and Bob take turns pocketing a coin from one of the ends (they choose each time, which end). (a) Prove: Alice can get at least half the value. (b) Prove: if we start with 101 coins, Alice *cannot* necessarily ensure that she gets at least as much money as Bob (even though she gets to pick one more coin than Bob), even if the value of each coin is either 1 or 2 units. (c) Suppose there are $2n+1$ coins, and the game master decides by flipping a fair coin $2n+1$ times which position gets a 1 and which gets a 2. Prove: for large n, it is very likely that Bob gets to pocket more than Alice. (d) Give a polynomial-time algorithm that gives each player their optimal move.