4.1 König Path Lemma revisited

Recall the statement of the König Path Lemma: Let $V = V_0 \cup V_1 \cup \cdots = \bigcup_{i=0}^{\infty} V_i$. Here $V_0 = \{s\}$, and have $V_i \cap V_j = \emptyset$ for $i \neq j$ (where $V_i$ denotes the $i$-th level of the tree).

Let us assume that all edges go between neighboring levels. More precisely, when $xy \in E$, we have some $i$ such that $x \in V_i$, $y \in V_{i+1}$. We then say that $x$ is a parent of $y$, and $y$ is a child of $x$.

Lemma 4.1.1. (König Path Lemma). Assume that

(0) $V_0 = \{s\}$ is the source,

(1) Every node has a finite number of children (which can be zero), and

(2) Every node on level $V_i$, for $i \geq 1$, has at least one parent.

Then, if $(\forall i)(V_i \neq \emptyset)$, then there is an infinite path going down (a “dynasty”).

Definition 4.1.2. $y$ is a descendant of $x$ if there is a path going down from $x$ to $y$.

Proof. We construct a dynasty $x_0, x_1, \ldots$ recursively ($x_i \in V_i$). The invariant property will be that $x_i$ has infinitely many descendants. This is true for the source, so let $x_0 = s$. Now if this is true for $x_i$ then it is also true for one of its children (because $x_i$ has only a finite number of children); let $x_{i+1}$ be the first child of $x_i$ who has infinitely many descendants.

We used the term “first child” which assumes an a priori ordering of each level. In most applications, such an ordering is evident (the nodes are named by strings over a finite alphabet, and we can order the strings lexicographically). If this is the case, this proof does not depend on the Axiom of Choice.

Without such a priori ordering, we need a countable version of the Axiom of Choice (called “Axiom of Dependent Choice”).

Let us explain some applications. Recall the following problem:
Problem 4.1.3. Given a finite set of (types of) tiles (all of size $1 \times 1$, and which must be tiled satisfying a particular rule: all tiles have a number on each side and those numbers must match up in adjacent tiles), the following are equivalent:

(A) The entire plane can be tiled,

(B) For all $n$, the $n \times n$-square can be tiled,

(C) The positive quadrant can be tiled.

Here, it is trivial that $(A) \Rightarrow (C) \Rightarrow (B)$. The big question is to show $(B) \Rightarrow (A)$. To do this, we apply the König Lemma: let the nodes on level $i$ be all tilings of the $(2i-1) \times (2i-1)$-square.

It is instructive to find a counterexample if we don’t have finitely many tiles:

Exercise 4.1.4. $B$ does not imply $A$ if we permit a countable number of tiles.

Next, recall the Erdős-DeBruijn Theorem:

Theorem 4.1.5. For $k \in \mathbb{N}$, a graph is $k$-colorable iff all finite subgraphs are $k$-colorable.

We shall now prove this in the special case when $G$ is countable.

Proof. Without loss of generality we may assume that $V = \mathbb{N}$. Let the set of colors be $\{0, \ldots, k-1\}$. To apply the König Path Lemma, let the level-$n$ nodes be all $k$-colorings of the induced subgraph $G\lvert_{\{0, \ldots, n-1\}}$; there are at most $k^n$ such colorings. Every node has exactly 1 parent (restriction) and $\leq k$ children (extensions). The rest of the proof is left as an exercise.

Exercise 4.1.6. Finish the above proof.

4.2 Canonical notation for ordinals

For every ordinal $\alpha$, define

$$\alpha = \{ \beta \mid \beta < \alpha \}. \quad (1)$$

(This is a transfinite recursion.) For example,

$$\begin{align*}
\emptyset &= \emptyset, \\
1 &= \{ \emptyset \} = \{ \emptyset \}, \\
2 &= \{ \emptyset, 1 \} = \{ \emptyset, \{ \emptyset \} \}, \\
3 &= \{ \emptyset, 1, 2 \} = \{ \emptyset, \{ \emptyset \}, \{ \emptyset, \{ \emptyset \} \} \}, \\
& \quad \vdots \\
\omega &= \{ \emptyset, 1, 2, \ldots \}. \quad (7)
\end{align*}$$

Note here that if we replace $\emptyset$ by $\{ \}$ then $k$ has $2^{k+1}$ braces.
Notation 4.2.1. We shall omit the underline: when writing $\alpha$, we shall mean the set $\alpha$. (For us, doing naive set theory, this is just a convention; in axiomatic set theory, this would be the definition of ordinals.)

With this notation, the following are equivalent for ordinals $\alpha, \beta$:

$$\beta < \alpha, \ \beta \in \alpha, \ \beta \text{ is a proper prefix of } \alpha, \ \beta \subset \alpha. \quad (8)$$

So, $\omega_1$, the smallest uncountable ordinal, is also the set of all countable ordinals.

### 4.3 Fodor’s Lemma

**Lemma 4.3.1.** For any $f : \omega_1 \to \omega_1$ which is regressive ($f(0) = 0$ and $(\forall \beta \neq 0)(f(\beta) < \beta)$), we have

$$\exists \alpha(|f^{-1}(\alpha)| = \aleph_1). \quad (9)$$

**Proof.** The idea of the proof is: iterate. If we iterate $f$, then starting with any ordinal, we must hit zero after finitely many steps (otherwise, we would have an infinite descending chain of ordinals, which cannot exist because every ordinal is well-ordered).

Now, let us assume, for sake of a contradiction, that $(\forall \alpha)(|f^{-1}(\alpha)| \leq \aleph_0)$. Let us partition the countable ordinals by how many steps it takes to get down to zero. Only zero can take zero steps to get down to zero. Then, there can be at most countably many that take one step to get down to zero (since $f^{-1}(0)$ is countable). Inductively, if there are at most countably many ordinals that take $n$ steps to get down to zero ($f^{-n}(0)$ is countable), then the only ordinals which can take $n + 1$ steps are those that map to one of the ordinals that take $n$ steps to reach zero ($f^{-(n+1)}(0) = f^{-1}(f^{-n}(0))$). This involves taking the union of the preimages of countably many ordinals; since each such preimage is countable, we have the countable union of countable sets, which must be countable. Hence, there must be countably many ordinals that take $n + 1$ steps. Since all ordinals take finitely many steps, the collection of all countable ordinals is a countable union of countable sets (the union for all $n$ of the ordinals that take $n$ steps), i.e., $\omega_1 = \bigcup_{n<\omega} f^{-n}(0)$, and must therefore be countable.

But, the set of all countable ordinals ($\omega_1$) is uncountable, a contradiction. \qed

### 4.4 The slot machine problem revisited

This solves the slot machine problem. Recall it:

**Problem 4.4.1.** Suppose that we have a “very generous” slot machine, which returns countably many coins for every coin that is inserted into the slot machine. However, it will never return a coin that you inserted. Show that, after countably many steps, you will lose all your money.

Note first the importance of the slot machine never returning a coin you placed in: without this “greed,” the set of coins you have after $\omega$ steps could be undefined. (Imagine there is a red coin and a blue coin; on odd numbered steps, you put the red coin in the machine...
and receive the blue coin; on even numbered steps you put in the blue coin and receive the red coin. Now after \( \omega \) steps, do you have the red or the blue coin?)

The greedy property permits us to give a precise (recursive) definition of the set of coins you have at the beginning of stage \( \alpha \):

\[
\text{coins}(\alpha) = \bigcup_{\beta < \alpha} \text{reward}(\beta) \setminus \{\text{spent}(\beta) : \beta < \alpha\}. \tag{10}
\]

Next we observe that it is possible to lose all your money in only \( \omega \) steps. Here is how. Say, the coins you receive in round \( i \) are labeled \( c_{i0}, c_{i1}, c_{i2}, \ldots \). Then, suppose your strategy is to play coins forming consecutive \( L \)'s:

\[
c_{00}, c_{01}, c_{11}, c_{10}, c_{02}, c_{12}, c_{22}, c_{21}, c_{20}, \ldots \tag{11}
\]

(the \( L \)'s are partitioned along \((c_{00}), (c_{01}, c_{11}, c_{10}), (c_{02}, c_{12}, c_{22}, c_{21}, c_{20}), etc.) It is then clear you will lose all your money in \( \omega \) steps.

Note that we can survive in this game to any countable ordinal: Pick a bijection between the given ordinal and \( \omega \), and then order the countable coins you receive at the first step according to that ordinal: then you can play according to that (only spending the coins you received on the first round) to the given ordinal without losing all your money (you will have all the money gained on subsequent rounds). However, you have to decide in advance what that ordinal is: it won’t just work indefinitely (for all countable ordinals). (The situation is somewhat like playing a game with another person to see who can name the larger number (or ordinal): the second player will always win, even though the first player can ensure their number is at least as big as any given number.)

**Exercise 4.4.2.** Solve Problem 4.4.1 using Fodor’s Lemma. (Where is the regressive function in the slot-machine game?)

What ordinals can possibly be the first step where we have no money left to play? Clearly, it cannot be on step \( \omega + 1 \), because if we have money left on step \( \omega \), then after putting a coin in we receive countably many back. More generally, such a step must be a limit ordinal.

**Exercise 4.4.3.** For any countable limit ordinal \( \alpha \), there exists a strategy where the time-step the player first has no money to play is \( \alpha \).

### 4.5 More puzzles on ordinals, transfinite recursion and transfinite induction

**Notation 4.5.1.** For all ordinals \( \alpha \), there is a cardinal \( \aleph_\alpha \). Then, we denote the smallest ordinal of cardinality \( \aleph_\alpha \) by \( \omega_\alpha \).

Note that an ordinal is \( \omega_\alpha \) for some \( \alpha \) iff all of its prefixes have smaller cardinality.

**Exercise 4.5.2.** There exists a subset \( S \subset \mathbb{R}^2 \) such that, for all lines \( \ell \subset \mathbb{R}^2 \), \(|S \cap \ell| = 2\).
Hint: use transfinite recursion to construct $S$.

**Exercise 4.5.3.** Suppose that $\alpha$ is not finite. Then

$$|\omega_\alpha \times \omega_\alpha| = |\omega_\alpha|,$$

(12)
i.e., $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$. Prove this using transfinite induction on $\alpha$.

We have the following conceivable ordinal distributivity properties, one of which is false, and one of which is true:

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

(13)

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma.$$  

(14)

**Exercise 4.5.4.** Which one is false, and which true, and why? Make your counterexample for the false one minimal.

## 4.6 Finitely additive $(0,1)$-measures.

**Definition 4.6.1.** For $S \neq \emptyset$ a set, a **finitely additive $(0,1)$-measure** is a function $\mu : \mathcal{P}(S) \to \{0, 1\}$, such that:

1. $\mu(\emptyset) = 0$,
2. $\mu(S) = 1$,
3. If $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Recall the following definition of a **dictator**: an element $s \in S$ such that the image of any subset $A \in \mathcal{P}(S)$ only depends on whether or not $s \in A$.

**Exercise 4.6.2.** If $S$ is finite, then such a $\mu$ has a dictator.

**Theorem 4.6.3.** If $S$ is infinite, then there exists a nontrivial $\mu$ (i.e., one that does not contain a dictator).

To solve this, we need the following:

**Definition 4.6.4.** A subset $\mathcal{A} \subset \mathcal{P}(S)$ is called a **filter** if

1. $\mathcal{A}$ is “upward closed,” i.e.,

$$(\forall A \in \mathcal{A})(\forall B \supseteq A)(B \in \mathcal{A}).$$

(15)

2. $\mathcal{A}$ is closed under finite intersections, i.e.,

$$(\forall A, B \in \mathcal{A})(A \cap B \in \mathcal{F}).$$

(16)
Definition 4.6.5. A subset \( \mathcal{A} \subseteq \mathcal{P}(S) \) is called an ideal if

1. \( \mathcal{A} \) is “downward closed”: for all \( B \subseteq A \in \mathcal{A} \), we have \( B \in \mathcal{A} \).
2. \( \mathcal{A} \) is closed under finite unions, i.e., for all \( A, B \in \mathcal{A} \), we have \( A \cup B \in \mathcal{A} \).

Definition 4.6.6. A filter \( \mathcal{F} \subset \mathcal{P}(S) \) is an ultrafilter if

1. \( \mathcal{F} \neq \emptyset \);
2. \( \mathcal{F} \neq \mathcal{P}(S) \);
3. \( \mathcal{P}(S) \setminus \mathcal{F} \) is an ideal.

Exercise 4.6.7. For a finitely additive \((0, 1)\)-measure, the set \( \mu^{-1}(1) \) is an ultrafilter.

In fact, ultrafilters and finitely additive \((0, 1)\)-measures are in 1-1 correspondence: any ultrafilter corresponds to \( \mu^{-1}(1) \) for a unique \( \mu \).

Exercise 4.6.8. Use Zorn’s lemma to construct a nontrivial ultrafilter on any infinite set.

Exercise 4.6.9. Use a nontrivial \((0, 1)\)-measure to solve \( \frac{1}{2} \) of a puzzle problem (you should know which one).