7.1 Matchings

Suppose we have a bipartite graph \( G = (V, E) \) with set of vertices \( B = A \cup \bar{B} \) (with all edges having one endpoint from \( A \) and one endpoint from \( B \)).

**Definition 7.1.1.** A matching \( A \rightarrow B \) is an injection satisfying

\[
(\forall x \in A)((x, f(x)) \in E).
\]

Our first main goal is to address the

**Question 7.1.2.** Given a finite bipartite graph, does there exist a matching?

This depends on the graph. Some graphs clearly do not admit such an injection: for example, one could have a König-Hall obstacle:

**Definition 7.1.3.** A König-Hall obstacle to an injection satisfying (1) is a set \( S \subseteq A \) such that \( |N(S)| < |S| \), where \( N(S) \) is the set of neighbors of \( S \).

**Theorem 7.1.4.** (König-Hall) If \( G \) is a finite bipartite graph as above, then a matching exists iff there is no König-Hall obstacle.

**Exercise 7.1.5.** Prove the above theorem.

**Exercise 7.1.6.** Prove that the König-Hall theorem is false for countably infinite graphs.

**Exercise 7.1.7.** Prove that König-Hall’s theorem is true for infinite bipartite graphs assuming that \((\forall x \in A)(\deg(A) < \aleph_0)\).

Note that the König-Hall theorem is a statement of the form \( \exists \Leftrightarrow \forall \). Such a statement is called a “good characterization,” since there is now information that can be easily verified to show whether or not something exists.

7.2 Erdős-Hajnal theorem

We now get to the highlight of this course:

**Theorem 7.2.1.** (Erdős-Hajnal) If \( \chi(G) > \aleph_0 \) then \((\forall m < \aleph_0)(G \supset K_{m,\aleph_1})\). That is, if \((\exists m)(G \not\supset K_{m,\aleph_1})\), then \( \chi(g) \leq \aleph_0 \).
For the theorem we will need the following general fact:

**Lemma 7.2.2.** If $G = (V,E)$ and subgraphs $G_1, G_2 \subset G$ with $G_1 = (V,A), G_2 = (V,B)$ satisfy $E = A \cup B$, then $\chi(G) \leq \chi(G_1) \cdot \chi(G_2)$.

**Proof.** If $C_1, C_2$ are colorings of $G_1, G_2$, then we obtain a coloring of $G$ by $C_1 \times C_2$ by placing at each vertex the colors from the colorings of $G_1, G_2$. It is easy to see that this is a coloring, and the lemma follows. \qed

We attempt to prove the theorem when $|V| = \aleph_1$. Without loss of generality, $V = \omega_1 = \{\alpha < \omega_1\} = \text{the set of countable ordinals}.$

We will use the reformulation of the statement of the theorem (if $(\exists m)(G \not\models K_{m,\aleph_1})$, then $\chi(g) \leq \aleph_0$). For concreteness, suppose $m = 7$, and let us assume that $G \not\models K_{7,\aleph_1}$.

**Definition 7.2.3.** Let $A \subseteq V$ and $x \in V \setminus A$. Then, $x$ is **troublesome** if $\#(\text{edges } x - A) \geq 7$.

**Claim 7.2.4.** If $|A| = \aleph_0$ then the number of troublesome vertices is $\leq \aleph_0$.

**Proof.** The point is that $(\aleph_0^7) \leq \aleph_0$ (because the LHS is $\leq \aleph_0^\aleph_0 = \aleph_0$). For each subset of $A$ of size $7$, there can be at most countably many vertices which are neighbors of the whole subset. \qed

Thus, we can take a subset $A \subset V$ of size $|A| = \aleph_0$, and add all the troublesome vertices to $A$. Call the resulting set $T(A) \supset A$.

Now, we iterate this process: Define

$$T^0(A) := A, \quad T^m(A) := T(T^{m-1}(A)) \text{ (for } 1 \leq M < \omega), \quad T^\omega(A) := \bigcup_{n<\omega} T^n(A).$$

(2)

**Claim 7.2.5.** $T(T^\omega(A)) = T^\omega(A)$. That is, $T^\omega(A)$ is closed under “trouble.”

**Proof.** If there existed an $x$ which is troublesome for $T^\omega(A)$, then it would have to be troublesome for $T^m(A)$ for some $m < \omega$, since we only need to find seven neighbors of $x$ in $T^\omega(A)$. But, then $x \in T^{m+1}(A)$, a contradiction. \qed

Now, remember that $V = \omega_1$. For $\alpha < \omega_1$, define $B_\alpha = T^\omega(A)$. Then,

$$\alpha < \beta \implies B_\alpha \subseteq B_\beta.$$ 

(3)

Next, let us pick orientations of the edges in $E$ so that all edges between $\omega_1 \setminus B_\alpha$ and $B_\alpha$ point towards $B_\alpha$, for all $\alpha$. (This is possible because of (3).)

$$N := \bigcup_{\alpha<\omega_1} \text{ (edges from } \omega_1 \setminus B_\alpha \text{ to } B_\alpha).$$

(4)

Also, define $M := E \setminus N$.

**Notation 7.2.6.** For any vertex, call $\deg^+(x)$ the “out degree” of $x$: the number of edges coming out of $x$. For any subgraph $G' \subset G$, let $\deg_{G'}$ and $\deg^+_{G'}$ denote the degrees with respect to $G'$.

Let $G_1 := (\omega_1, N)$. Then, we make the

**Claim 7.2.7.** $(\forall x \in \omega_1)(\deg^+_{G_1}(x) \leq 6)$.
Proof. Each $B_\alpha$ is trouble-free, so there can be at most 6 edges from any $x \notin B_\alpha$ to $B_\alpha$, which gives the desired result.

Now, we have the essential

**Claim 7.2.8.** $\chi(G_1) \leq 13 = 2 \cdot 6 + 1$.

Using the claim, let us finish the proof. We need only show that $\chi(\omega_1, M) \leq \aleph_0$; then Lemma 7.2.2 gives the desired result.

Let $G_2$ be all the graph $(V, M)$ of all edges not in $G_1$. In $M_1$ there is no ede between $B_\alpha$ and $\omega \setminus B_\alpha$. Then, each connected component of $G_2$ is countable because the connected component of vertex $\alpha$ is contained in $B_{\alpha+1}$. So it must be colorable by countably many colors, and we’re done.

**Exercise 7.2.9.** Extend this proof to all cardinals. Hint: if $|V| = \aleph_\gamma$ for $\gamma \geq 1$, then we can set $V := \omega_\gamma$, and perform transfinite induction on $\gamma$.

Let us start the proof of Claim 7.2.8. We first prove the following result: if $(\forall x)(\deg(x) \leq k)$ then $\chi(G) \leq k+1$. To prove this, we define a coloring by transfinite recursion, $f : \alpha \rightarrow k+1 = \{0, \ldots, k\}$. Suppose that $f(\gamma)$ has been defined for all $\gamma < \beta$. Then, set $f(\beta) := \min((k+1) \setminus \{f(\gamma) : \gamma < \beta, \gamma \sim \beta\})$. We also have the following exercise from last time:

**Exercise 7.2.10.** Let $\kappa \geq \aleph_0$. If $(\forall x)(\deg(x) \leq \kappa)$, then $\chi(G) \leq \kappa$. Hint: connected components. (Show that every connected component has cardinality at most $\kappa$.)

Now, we prove the following, which was an exercise:

**Theorem 7.2.11.** If $G$ is a digraph and $(\forall x)(\deg^+(x) \leq k)$, then $\chi(G) \leq 2k + 1$.

To prove it, we first prove it for finite graphs, and then apply Erdős-DeBruijn (Theorem 4.15). As warm-up, let’s prove the

**Theorem 7.2.12.** Any finite planar graph is 6-colorable.

**Proof.** We need the following

**Lemma 7.2.13.** If $G$ is finite and planar, then $(\exists x)(\deg(x) \leq 5)$.

**Exercise 7.2.14.** Prove the above lemma with Euler’s formula.

Now, we can simply remove such a vertex, apply a six-coloring (which we assume exists by induction); then since that vertex had only five neighbors, there must be a color we can assign it.

**Exercise 7.2.15.** Using all of the above, prove Theorem 7.2.11.
7.3 \(2^{\aleph_0} \rightarrow (\aleph_1, \aleph_1)\)

Recall the

Exercise 7.3.1. \(2^{\aleph_0} \rightarrow (\aleph_1, \aleph_1)\).

Solution 7.3.2. Define a well-ordering of the real numbers \((\mathbb{R}, \prec)\). Let \((\mathbb{R}, <)\) be the natural ordering. We need to color the edges of \(K_\mathbb{R}\) red and blue such that all homogeneous subsets are countable. Let us set

\[
\text{color}\{r, s\} = \begin{cases} 
\text{red}, & \text{if either } r \prec s \text{ and } r < s, \text{ or if } s \prec r \text{ and } s < r \text{ (i.e., the two orderings agree)}; \\
\text{blue}, & \text{otherwise.}
\end{cases}
\]

(5)

Now, suppose that \(A \subseteq \mathbb{R}\). \(A\) is all red implies that \(A\) is well-ordered in \(\mathbb{R}\). We have the

Lemma 7.3.3. If \(A\) is a well-ordered subset of \(\mathbb{R}\), then \(A\) is countable.

This lemma solves the exercise, since we can equally apply it when reversing the natural order of \(\mathbb{R}\).

Lemma. For every element \(a \in A\), we can assign any rational number which lies strictly between \(a\) and its successor. This gives an injection \(A \hookrightarrow \mathbb{Q}\), so \(A\) is countable. \(\Box\)

7.4 Countably additive \((0, 1)\)-measures revisited.

Suppose that there exists a \(\sigma\)-additive \((\text{:= countably-additive})\) \((0, 1)\)-measure \(\mu\) on a set \(A\) with \(|A| = \kappa\). Assume \(\kappa\) is the smallest such cardinal. We wish to prove:

(a) \(\mu\) is \(< \kappa\)-additive, i.e.,
\[
\mu\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mu(A_i),
\]

for all \(|I| < \kappa\).

(b) If \(\lambda < \kappa\), then \(2^\lambda < \kappa\). (In particular, \(\kappa > \tau = 2^{\aleph_0} > 2^{2^{\aleph_0}} > \cdots\))

(c) \(\kappa\) is nonsingular.

(d) \(\kappa \rightarrow (\kappa, \kappa)\).

Definition 7.4.1. \(\kappa\) is strongly inaccessible if (b) and (c) hold.

Definition 7.4.2. \(\kappa\) is a measurable cardinal if \(\exists\ (< \kappa)\)-additive nontrivial \((0, 1)\)-measure on \(\kappa\).

The existence of measurable cardinals is often taken as an assumption, although one cannot prove the consistency of them with set theory (it has been proved that any such proof would make set theory inconsistent).

In fact, the existence of measurable cardinals would contradict the \(V = L\) axiom (that every set is constructible). This is related to Gödel’s proof of the consistency of the generalized continuum hypothesis (GCH), since Gödel used the constructible sets as his model of ZFC+GCH.