8.1 Some Erdős-Rado Arrows

Exercise 8.1.1. 

• Prove $\alpha \rightarrow (\aleph_1, \aleph_1)$.

• Let $\alpha^+$ denote the successor function on cardinals, i.e., if $\alpha$ is a cardinal, $\alpha^+$ is the least cardinal greater than $\alpha$. Prove $\alpha^+ \rightarrow (\aleph_1, \aleph_1)$.

• Let $\alpha$ be an infinite cardinal. Prove $(2^{\alpha})^+ \rightarrow (\alpha^+, \alpha^+)$. 

• (Sierpinski) Prove if $\alpha$ is an infinite cardinal then $2^\alpha \nrightarrow (\alpha^+, \alpha^+)$. 

Exercise 8.1.2 (Sierpinski). Assume the Continuum Hypothesis (CH). Prove there exists a partition $\mathbb{R}^2 = A \sqcup B$ such that $A \cap \ell_h, B \cap \ell_v$ are countable for every horizontal line $\ell_h$ and every vertical line $\ell_v$.

8.2 Random Graphs

Let $X$ and $Y$ be two random graphs with $n$ vertices. Then $\text{Prob}(X = Y) = 2^{-\binom{n}{2}}$. We then have, 

$$\text{Prob}(X \cong Y) \leq \sum_{\sigma \in S_n} \text{Prob}(\sigma(X) = Y) = \frac{n!}{2^{\binom{n}{2}}} < \left(\frac{n}{2^{(n-1)/2}}\right)^n.$$ 

This probability goes to zero very fast. Suppose now that the vertex set is $\omega$.

Theorem 8.2.1 (Erdős-Rényi). $\text{Prob}(X \cong Y) = 1$ if $X$ and $Y$ are random graphs with vertex set $\omega$.

Let us motivate the proof. Number the vertices as $0, 1, 2, \ldots$. What is the probability of finding a vertex $v$ satisfying a given adjacency relation with vertices, say, $0, 1, 2, 3, 4$? The probability that vertex $i > 4$ satisfies the desired relations is $2^{-5}$. Hence the probability that no vertex $i$ between 5 and $4+n$ satisfies the relation is $(1 - 2^{-5})^n$. Hence the probability of finding a vertex $v$ as desired is 1. Similarly, the probability is 1 given any finite number of vertices and adjacency relations.

Proof. Let $k < \infty$ and $\epsilon: k \to 2 = \{0, 1\}$. Let 

$$A_{x, \epsilon} : (\forall x_0, \ldots, x_{k-1})(\text{if } x_i \text{ are distinct then } (\exists y)(\bigwedge_{i=0}^{k-1} R(y, x_i) = \epsilon(i)))$$

Since there are only countably many $A_{x, \epsilon}$’s we see that a random graph satisfies all of them with probability 1. The following exercise concludes the proof. 

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Exercise 8.2.2. Show that this system of axioms is categorical in \(\aleph_0\). Hint: Let \(A\) and \(A'\) be two models. Number the elements from 0. Let \(0 \mapsto 0'\). Let \(v \mapsto 1'\) where \(v\) satisfies the same adjacency relation with 0 as \(1'\) does with \(0'\). Let \(w\) be the least element that hasn’t been mapped. Let \(w \mapsto w'\) where \(w'\) satisfies the same adjacency relation with \(0', 1'\) as \(w\) does with \(0, v\). Keep going back and forth and show we get a bijection.

We shall refer to the unique countable model of this set of axioms at “the Random Graph.”

Exercise 8.2.3. Prove: the Random Graph is universal in the sense that every countable graph is an induced subgraph of it.

Exercise 8.2.4. Prove that the number of isomorphism classes of countable graphs is \(2^{\aleph_0}\).

Hence even though two countable random graphs are almost always isomorphic to each other, there are a lot of distinct isomorphism classes of countable graphs.

8.3 Axiomatisability

Here we consider some graph-theoretic notions and see whether we can axiomatise them using first order logic with equality and adjacency relation and whether we can do it with finitely many of them. Since we can AND finitely many statements it is clear that finite number of axioms is equivalent to a single axiom. Since we have an equality relation, we also have the \(k\)-ary relation of distinctness for any \(x_1, \ldots, x_k\).

We have seen above that the Random Graphs is axiomatisable by countably many axioms. Below is a list of notions and axioms for them.

- Complete graphs: \((\forall x, y)(\text{if } x \neq y \text{ then } R(x, y) = 1)\).
- Triangle-free graphs: \((\forall x, y, z)(\text{if } x, y, z \text{ are distinct then } \neg R(x, y) \lor \neg R(x, z) \lor \neg R(y, z))\)
- Graphs whose complement is triangle-free: \((\forall x, y, z)(\text{if } x, y, z \text{ are distinct and } (\neg R(x, z) \land \neg R(y, z)) \text{ then } R(x, y))\)
- Complete multipartite graphs: the nonadjacency is a transitive relation.
- Bipartite graphs: no 3-cycle, no 5-cycle, \ldots (no odd cycles): this is a countable system of axioms.
- Graphs of diameter \(\leq 10\): \((\forall x, y)(\exists z_0, \ldots, z_{10})(x = z_0 \land y = z_{10} \land (\land_{i=0}^{9}(z_i = z_{i+1} \lor R(z_i, z_{i+1}))))\)
- Let \(P_n(x, y)\) be the formula that \(\exists\) a path of length \(\leq n\) between \(x\) and \(y\). Graphs with infinite diameter are axiomatisable by countably many axioms: \((\exists x, y)(\neg P_n(x, y))\). Note that \(\mathbb{N}\) with its natural adjacency relation (the one-way infinite path) is a connected infinite graph of infinite diameter.

Question 8.3.1. Can we axiomatise bipartite graphs with finitely many axioms?

Theorem 8.3.2. Connected graphs, disconnected graphs are not axiomatisable. Bipartite graphs, 3-colorable graphs are axiomatisable but not finitely axiomatisable.

To prove the claims of nonaxiomatizability we need the construction of ultraproducts.
8.4 Ultraproducts and Axiomatisability

Suppose $A_i, i \in I$ are models of the same language. Then $\prod_{i \in I} A_i$ is also a model for the same language. Take graphs for example. If $f, g \in \prod_{i \in I} A_i$ we can define adjacency relation between them as $(R(f, g) \text{ if } (\forall i \in I)(R(f(i), g(i))))$.

Suppose now that $\mu$ is a finitely additive $(0, 1)$-measure on $I$. Define an equivalence relation on $\prod_{i \in I} A_i$ by $f \sim g$ if $f(i) = g(i)$ almost always, i.e., $\mu(i | f(i) = g(i)) = 1$. Let $\prod A_i/\mu$ denote the set of equivalence classes. Then this is called the ultraproduct of the $A_i$ with respect to $\mu$ and this is also a model for the same language. Again for graphs we can define adjacency relation between equivalences classes $[f], [g]$ by $(R([f], [g]) \text{ if } R(f(i), g(i)) \text{ a.a.})$.

**Theorem 8.4.1 (Lósa (abridged)).** A sentence $\varphi$ is true in $U = \prod A_i/\mu$ iff $\varphi$ is true in $A_i$ a.a

**Theorem 8.4.2 (Lósa (unabridged)).** Let $\varphi$ be a formula and $f_1, \ldots \in U$. Then $\varphi(f_1, \ldots)$ is true in $U$ iff $\varphi(f_1(i), \ldots)$ is true in $A_i$ a.a.

A proof can be given by inducting on number of steps in the construction of the formula from primitive formulae.

**Exercise 8.4.3.** Let $H = \prod_p \mathbb{F}_p/\mu$ where $\mathbb{F}_p$ is the field with $p$ elements and $\mu$ is not a dictator measure. $|H| = c$. What is its characteristic? (Hint: From Lósa’s theorem it cannot be any prime)

**Connectedness cannot be axiomatised.** Consider the ultraproduct $U$ of cycles of all lengths $n \in \mathbb{N}$, $n \geq 3$. It is a graph with $c$ many points. Now vertices with degree 2 can be axiomatised and all vertices in a cycle have degree 2. Hence by Lósa’s theorem every vertex of $U$ also has degree 2. We have seen that every connected component of a graph whose vertices have only degree 2 is either a 2-way infinite path or a finite cycle. However, by Lósa’s theorem, no connected component of $U$ can be a finite cycle. Since there are $c$ many points in $U$ it has to have $c$ many connected components. Hence we have obtained a disconnected graph after taking the ultraproduct of connected graphs. Hence connected graphs cannot be axiomatised using first order logic with equality and adjacency relation.

**Disconnectedness cannot be axiomatised.** The proof above also shows that disconnectedness cannot be axiomatised with finitely many axioms. (If not, we can OR the negation of all these axioms and we get an axiomatisation of connectedness). We prove that infinitely many axioms also do not suffice.

Suppose $A_i, i \in I$ is a set of axioms for disconnectedness. Let $U = \prod_{\aleph_0}(2\text{-way infinite path})/\mu$. It is the same $U$ as in the proof above. Now one of the axioms, say $A_j$, fails for the 2-way infinite path. Hence $\neg A_j$ is true for it and hence for $U$, a contradiction.

**Bipartiteness cannot be finitely axiomatised.** We prove, in fact, that non-bipartiteness cannot be axiomatised. Every odd cycle is non-bipartite. Take the ultraproduct of all odd cycles. We get $U$ again which is bipartite.

**Exercise 8.4.4.** Non 3-colorability is not axiomatisable. Hence 3-colorability is not finitely axiomatisable

**Exercise 8.4.5.** Use ultraproducts to prove the Compactness Theorem of First Order Logic
Axiomatisation of $\mathbb{N}$ cannot capture the notion of every $n \in \mathbb{N}$ being reachable from 0 by applying the successor function finitely many times. This is the case since the ultraproduct of countably many copies of $\mathbb{N}$ is uncountable and is also a model for first order arithmetic. However, the set of elements reachable from 0 by applying the successor function finitely many times is countable.

This ultraproduct is an example of a “non-standard model of arithmetic.”

Notice that primes in this ultraproduct will be those functions which take prime values almost always. First-order statements like “every integer is a sum of 4 squares” or “a prime $p$ is a sum of two squares if and only if $p = 2$ or $p \equiv 1 \pmod{4}$” remain valid in this model of arithmetic. In fact from Lós’ theorem, all first order statements are valid in this uncountable model.