

BABAI DISCRETE MATHEMATICS REU 2008
EXERCISES FROM LECTURE 1, MON JUNE 23

SCRIBE: JOSHUA A. GROCHOW

Definition (Arithmetic progression). An **arithmetic progression** is a sequence of the form $a, a + d, a + 2d, a + 3d, \dots$. d is known as the **increment** of the arithmetic progression.

Exercise 1. Prove that for any number n not divisible by 7, the sequence $0, n, 2n, 3n, \dots, 6n$ represents each residue modulo 7 exactly once.

Exercise 2. Prove that a 7-term arithmetic progression of primes whose smallest term is greater than 7 must have increment divisible by 210.

Exercise 3. More generally, prove that a k -term arithmetic progression of primes whose first term is greater than k must have increment divisible by every prime $p \leq k$.

Theorem (Prime Number Theorem). The prime counting function $\pi(x) = \#\{p : p \text{ is prime and } p \leq x\}$ satisfies

$$\pi(x) \sim \frac{x}{\ln(x)}$$

Definition (Asymptotic equality). Two sequences a_0, a_1, a_2, \dots and b_0, b_1, b_2, \dots are **asymptotically equal**, denoted $a_n \sim b_n$, if

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1.$$

(For the purposes of this definition, if $a_k = b_k = 0$, we take a_k/b_k to be 1.)

Exercise 4. Prove that the following statement is equivalent to the Prime Number Theorem:

$$\prod_{p \leq x} p \sim e^{x(1+\varepsilon_x)}$$

where the product is over primes $\leq x$, and $\lim_{x \rightarrow \infty} \varepsilon_x = 0$.

Exercise 5. Prove that for any fixed $c > 0$, (i.e. c does not depend on x) $\lim_{x \rightarrow \infty} \frac{\ln(x)}{x^c} = 0$.

Exercise 6. Prove that e is irrational. *Hint:* $e = \sum_{n=1}^{\infty} \frac{1}{n!}$.

Exercise 7 (Lemma 2 from class). Prove that if p is prime and $p^t \mid \binom{n}{k}$ then $p^t \leq n$.

***Exercise 8.** Prove that if the natural numbers are partitioned into $n \geq 2$ disjoint arithmetic progressions $\{a_1 + kd_1 : k \in \mathbb{N}\}, \{a_2 + kd_2 : k \in \mathbb{N}\}, \dots, \{a_n + kd_n : k \in \mathbb{N}\}$ then at least two of the increments d_1, \dots, d_n must be equal.

Exercise 9. Suppose a regular n -gon with vertices a_0, a_1, \dots, a_{n-1} is inscribed in the unit circle. Prove that the product of the lengths of the lines $\overline{a_0a_1}, \overline{a_0a_2}, \overline{a_0a_3}, \dots, \overline{a_0a_{n-1}}$ is equal to n .

Exercise 10. Formulate a conjecture about which primes can be written as the sum of two squares.

- (1) Experiment with small primes, observe the simple pattern, and make a conjecture;
- (2) prove that if a prime does not satisfy your hypothesis then it cannot be written as a sum of two squares (should be relatively easy); and
- (3) ** [Fermat] prove that if a prime does satisfy your hypothesis then it can be written as a sum of two squares.

Definition. Given a set S of n distinct points in the plane, let $u(S)$ denote the number of pairs of these points that are exactly distance 1 apart, i. e., $u(S) = \#\{\{x, y\} : x, y \in S \text{ and } \text{dist}(x, y) = 1\}$. Let $m(n) = \max\{u(S) : |S| = n\}$.

Exercise 11. (1) Prove there is a constant c such that for all n , $m(n) < cn^{3/2}$. *Hint:* graph theory.

- (2) Prove that $\lim_{n \rightarrow \infty} \frac{m(n)}{n} = \infty$.
- (3) Prove that for any fixed k ,

$$\lim_{n \rightarrow \infty} \frac{m(n)}{n(\ln n)^k} = \infty$$

Hint: number theory.

Definition (Tournament). A **tournament** is an orientation of a complete graph, i. e., it consists of n vertices, and for each pair of distinct vertices $x \neq y$, exactly one of x or y is chosen as the “winner.” A tournament is **k -paradoxical** if for every set S of k vertices, there is a vertex that beats every vertex in S .

Exercise 12. Prove that for every k , there exists a k -paradoxical tournament.

Exercise 13. Let $P_k(n)$ denote the fraction of tournaments on a given set of n vertices that are k -paradoxical. Prove that for any fixed k , $\lim_{n \rightarrow \infty} P_k(n) = 1$. (We say “A typical tournament is k -paradoxical,” or “A random tournament is almost surely k -paradoxical.”)

Exercise 14. Make a plausible suggestion of an explicit construction of a k -paradoxical tournament for any k . *Hint:* number theory.