Definition (Arithmetic progression). An arithmetic progression is a sequence of the form $a, a+d, a+2d, a+3d, \ldots$. $d$ is known as the increment of the arithmetic progression.

Exercise 1. Prove that for any number $n$ not divisible by 7, the sequence $0, n, 2n, 3n, \ldots, 6n$ represents each residue modulo 7 exactly once.

Exercise 2. Prove that a 7-term arithmetic progression of primes whose smallest term is greater than 7 must have increment divisible by 210.

Exercise 3. More generally, prove that a $k$-term arithmetic progression of primes whose first term is greater than $k$ must have increment divisible by every prime $p \leq k$.

Theorem (Prime Number Theorem). The prime counting function $\pi(x) = \# \{p: p$ is prime and $p \leq x\}$ satisfies

$$\pi(x) \sim \frac{x}{\ln(x)}$$

Definition (Asymptotic equality). Two sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are asymptotically equal, denoted $a_n \sim b_n$, if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$  

(For the purposes of this definition, if $a_k = b_k = 0$, we take $a_k/b_k$ to be 1.)

Exercise 4. Prove that the following statement is equivalent to the Prime Number Theorem:

$$\prod_{p \leq x} p \sim e^{x(1+\epsilon_x)}$$

where the product is over primes $\leq x$, and $\lim_{x \to \infty} \epsilon_x = 0$.

Exercise 5. Prove that for any fixed $c > 0$, (i.e. $c$ does not depend on $x$)

$$\lim_{x \to \infty} \frac{\ln(x)}{x^c} = 0.$$  

Exercise 6. Prove that $e$ is irrational. $\text{Hint: } e = \sum_{n=1}^{\infty} \frac{1}{n!}$.

Exercise 7 (Lemma 2 from class). Prove that if $p$ is prime and $p^j | \binom{n}{k}$ then $p^j \leq n$.  

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Exercise 8. Prove that if the natural numbers are partitioned into \( n \geq 2 \) disjoint arithmetic progressions \( \{a_1 + kd_1 : k \in \mathbb{N}\}, \{a_2 + kd_2 : k \in \mathbb{N}\}, \ldots, \{a_n + kd_n : k \in \mathbb{N}\} \) then at least two of the increments \( d_1, \ldots, d_n \) must be equal.

Exercise 9. Suppose a regular \( n \)-gon with vertices \( a_0, a_1, \ldots, a_{n-1} \) is inscribed in the unit circle. Prove that the product of the lengths of the lines \( a_0a_1, a_0a_2, a_0a_3, \ldots, a_0a_{n-1} \) is equal to \( n \).

Exercise 10. Formulate a conjecture about which primes can be written as the sum of two squares.

1. Experiment with small primes, observe the simple pattern, and make a conjecture;
2. Prove that if a prime does not satisfy your hypothesis then it cannot be written as a sum of two squares (should be relatively easy); and
3. **[Fermat]** prove that if a prime does satisfy your hypothesis then it can be written as a sum of two squares.

Definition. Given a set \( S \) of \( n \) distinct points in the plane, let \( u(S) \) denote the number of pairs of these points that are exactly distance 1 apart, i.e., \( u(S) = \#\{\{x, y\} : x, y \in S \text{ and } \text{dist}(x, y) = 1\} \). Let \( m(n) = \max\{u(S) : |S| = n\} \).

Exercise 11. (1) Prove there is a constant \( c \) such that for all \( n \), \( m(n) < cn^{3/2} \). **Hint:** graph theory.

2. Prove that \( \lim_{n \to \infty} \frac{m(n)}{n} = \infty \).

3. Prove that for any fixed \( k \),

\[
\lim_{n \to \infty} \frac{m(n)}{n(\ln n)^k} = \infty
\]

**Hint:** number theory.

Definition (Tournament). A **tournament** is an orientation of a complete graph, i.e., it consists of \( n \) vertices, and for each pair of distinct vertices \( x \neq y \), exactly one of \( x \) or \( y \) is chosen as the “winner.” A tournament is \( k \)-**paradoxical** if for every set \( S \) of \( k \) vertices, there is a vertex that beats every vertex in \( S \).

Exercise 12. Prove that for every \( k \), there exists a \( k \)-paradoxical tournament.

Exercise 13. Let \( P_k(n) \) denote the fraction of tournaments on a given set of \( n \) vertices that are \( k \)-paradoxical. Prove that for any fixed \( k \), \( \lim_{n \to \infty} P_k(n) = 1 \). (We say “A typical tournament is \( k \)-paradoxical,” or “A random tournament is almost surely \( k \)-paradoxical.”)

Exercise 14. Make a plausible suggestion of an explicit construction of a \( k \)-paradoxical tournament for any \( k \). **Hint:** number theory.