Theorem. If \( \mathbb{N} = A_1 \cup \cdots \cup A_k \) is the union of \( k \geq 2 \) disjoint arithmetic progressions, then not all of the increments are distinct.

Definition (Primitive root of unity). A complex number \( z \in \mathbb{C}^* \) is a primitive \( n \)-th root of unity if the order of \( z \) in \( \mathbb{C}^* \) is \( n \).

Theorem. Let \( q_1, \ldots, q_k \) be prime numbers congruent to 1 modulo 4. The number of ways to represent \( q_1 \cdots q_k \) as a sum of squares is equal to \( 2^{k-1} \) (where order does not matter, e.g. \( 5 = 1 + 2^2 = 2^2 + 1 \) counts as one way).

Theorem. Every prime in \( \mathbb{Z}[i] \) either has prime norm or is a rational prime congruent to 3 modulo 4.

Exercise 1. If \( q \) is a rational prime congruent to 1 modulo 4, then its factorization \( q = w\bar{w} \) in \( \mathbb{Z}[i] \) is such that \( w \neq u\bar{w} \) for any unit \( u \in \mathbb{Z}[i] \).

Theorem. Let \( n = q_1 \cdots q_k \) where the \( q_i \) are distinct primes congruent to 1 modulo 4. Consider a graph \( G \) whose vertices are the points of a \( 2\sqrt{n} \times 2\sqrt{n} \) grid in the plane, where two points are adjacent if the distance between them is exactly \( \sqrt{n} \). Then every vertex has degree at least \( 2^k \).

Theorem. If \( \pi(x) \) is the number of primes less than \( x \), then \( \pi(x) \sim x/\ln(x) \).
If \( \pi_{1(4)}(x) \) is the number of primes less than \( x \) and congruent to 1 modulo 4, then \( \pi_{1(4)}(x) \sim x/2\ln(x) \).

Definition. Say that \( f(n) \approx c^n \) if \( f(n) \) is dominated by \( b^n \) for all \( b > c \) and \( f(n) \) dominates \( b^n \) for all \( b < c \).

Exercise 2. Prove that
\[
\prod_{p < x} p \approx e^x \quad \text{and} \quad \left( \prod_{p < x, p \equiv 1(4)} p \right) \approx e^{x/2}.
\]

Theorem. If \( m(n) \) is the maximal number of unit distances among \( n \) points in the plane then \( m(n) = O(n^{3/2}) \). (Erdős conjectured that \( m(n) < n^{1+\epsilon} \) for any \( \epsilon > 0 \).)

Theorem. For \( d_1, \ldots, d_n \in \mathbb{R} \),
\[
\sqrt{\frac{\sum d_i^2}{n}} \geq \frac{\sum d_i}{n}.
\]
Exercise 3. Prove that there is a constant $c > 0$ and a family of graphs \( \{G_n\}_n \) without 4-cycles such that $G_n$ has $n$ vertices and at least $cn^{3/2}$ edges.

Definition (Set partition; Bell number). Define the Bell number $B_n$ to be the number of partitions of a set with $n$ elements, where a partition of the set $X$ is a collection of disjoint subsets $C_i \subseteq X$ ($i = 1, \ldots, j$) such that $X = C_1 \cup \cdots \cup C_j$.

Theorem. $B_n < n^n$. (Proved by inducing a partition from a function \( \{1, \ldots, n\} \to \{1, \ldots, n\} \), by taking the inverse images of elements.)

Exercise 4. Prove that $\ln(B_n) \sim n \ln(n)$.

Exercise 5. Prove the following:

1. $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$.
2. $\sum_{n=0}^{\infty} x^n B_n / n! = e^{e^x-1}$.
3. $B_n = e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!}$.

Hint: Prove the statements in the order given.

Theorem. $B_n \sim \frac{1}{\sqrt{n}} \lambda(n)^{\lambda(n)+1/2} e^{\lambda(n)-n-1}$ where $\lambda(n) \ln(\lambda(n)) = n$

Exercise 6. Find the largest term of $k^n / k!$ as $k$ varies, and prove that it is greater than $\lambda(n)^{\lambda(n)-n-1}$.

Exercise 7. The following is the outline of a proof that $\pi$ is irrational, due to Ivan Niven (1947). Prove that following:

1. If $f(x)$ is a polynomial, then $\int f(x) \sin(x) = F'(x) \sin(x) - F(x) \cos(x)$ where $F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots$ and $f^{(k)}(x)$ denotes the $k$-th derivative of $f$.
2. If $f_n(x) = x^n(a-bx)^n$ for $a, b \in \mathbb{Z}$, then for all $k$, $f_n^{(k)}(0) \in \mathbb{Z}$.
3. Use (1) and (2) to prove that $\pi$ is irrational.

Sketch for (3): Suppose that $\pi = a/b$. Use (2) to prove that for all $k$, $f_n^{(k)}(\pi) \in \mathbb{Z}$. Use (1) to prove that $0 < \int_0^\pi f_n(x) \sin(x) \in \mathbb{Z}$. On the other hand, prove that $\lim_{n \to \infty} \int_0^\pi f_n(x) = 0$. This is a contradiction.