Recall that we are considering only finite abelian groups.

**Definition.** A **character** $\chi$ of an abelian group $G$ is a group homomorphism $\chi : G \to T$, where $T = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

**Definition.** The $j$-th character $\chi_j$ of $\mathbb{Z}/n\mathbb{Z}$ is given by $\chi_j(m) = \omega^{mj}$ where $\omega_n = e^{2\pi i/n}$.

**Definition.** The **product character** for $\chi_1 : H \to T$ and $\chi_2 : K \to T$ is $\chi_1 \chi_2(h, k) = \chi_1(h) \chi_2(k)$.

**Theorem.** The characters of $G$ are an orthonormal basis for $\mathbb{C}G$, the $\mathbb{C}$-vector space of functions $G \to \mathbb{C}$ with the Hermitian inner product

$$ (f, h) = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} h(g). $$

So in particular there are $|G|$ characters of $G$.

**Definition.** For a finite abelian group $G$, the **dual group** $\hat{G}$ is the set of characters of $G$, with the group operation on $\hat{G}$ given by $(\chi \eta)(g) = \chi(g) \eta(g)$.

**Exercise 1.** Prove that $G \cong \hat{G}$. **Hint:** first prove for cyclic groups and then use structure theorem for finite abelian groups.

**Exercise 2.** Prove that a finite abelian group $G$ is canonically isomorphic to $\hat{\hat{G}}$ under the map $g \mapsto f_g$ where $f_g(\chi) = \chi(g)$.

**Theorem.** If $f \in \mathbb{C}G$, then $f = \sum_{\chi \in \hat{G}} c_\chi \chi$ where $c_{\chi} = (\chi, f)$.

**Definition.** The **Fourier transform** is a linear map $F : \mathbb{C}G \to \mathbb{C}^{\hat{G}}$ such that $F(f) = \hat{f}$ where

$$ \hat{f}(\chi) = n(\chi, f) = \sum_g \overline{\chi(g)} f(g) = \sum_g \chi(-g) f(g). $$

In matrix form, using the basis $\hat{G}$ for $\mathbb{C}G$ and basis $G = \hat{\hat{G}}$ for $\mathbb{C}^{\hat{G}}$, the Fourier transform is $C = (\chi(-g))$, where the rows are indexed by characters and the columns by group elements. This matrix is called the **character table for** $G$. 

Definition. A unitary transformation is a linear transformation \( \phi : V \to V \) such that \( (x, y) = (\phi x, \phi y) \), where \( V \) is a \( \mathbb{C} \)-vector space with inner product \( (\cdot, \cdot) \). In matrix form, this is equivalent to \( T^* T = TT^* = I \), where \( T^* \) is conjugate-transpose.

Theorem. If \( C \) is the character table for \( G \), then \( |G|^{-1/2} C \) is a unitary matrix. This statement is equivalent to the statement that the characters are orthonormal in the \( \mathbb{C}G \) inner product.

Theorem (First Orthogonality Relation). Let \( \chi_1 \) and \( \chi_2 \) be characters of a group \( G \). Then
\[
\frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g) = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2 \\ 1 & \text{if } \chi_1 = \chi_2 \end{cases}.
\]

Theorem (Second Orthogonality Relation). For \( g_1, g_2 \in G \),
\[
\frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(g_1)\chi(g_2) = \begin{cases} 0 & \text{if } g_1 \neq g_2 \\ 1 & \text{if } g_1 = g_2 \end{cases}.
\]

Definition. The Fourier inversion is a linear map such that \( I : \mathbb{C}\hat{G} \to \mathbb{C}G \) where
\[
I(h)(g) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(-g)h(\chi).
\]

Theorem. The Fourier transform and the Fourier inversion are inverses of each other, so \( I(F(f)) = f \) if \( f \in \mathbb{C}G \) and \( F(I(h)) = h \) if \( h \in \mathbb{C}\hat{G} \).

Theorem (Plancherel’s Formula). For \( f_1, f_2 \in \mathbb{C}G \), we have that
\[
(\hat{f}_1, \hat{f}_2) = |G| (f_1, f_2) \quad \text{and therefore} \quad \|\hat{f}_1\| = \sqrt{|G|}\|f_1\|.
\]

Definition. For \( A \subset G \) define the characteristic function \( f_A : G \to \mathbb{C} \) by
\[
f_A(g) = \begin{cases} 0 & \text{if } g \notin A \\ 1 & \text{if } g \in A \end{cases}.
\]

Theorem. For \( A \subset G \), if \( \chi_0 \) is the principal character, then \( \hat{f}_A(\chi_0) = |A| \).

Exercise 3. If \( \chi \neq \chi_0 \), then \( \hat{f}_A(\chi) = \hat{f}_{G \setminus A}(\chi) \).

Theorem. \( |\hat{f}_A(\chi)| \leq |A| \). (Use the definition of Fourier transform and triangle inequality.)

Definition. If \( A \subset G \), define \( \Phi(A) = \max_{\chi \neq \chi_0} |\hat{f}_A(\chi)| \).

Exercise 4. If \( A \subset G \) with \( |A| \leq |G|/2 \), then \( \sqrt{|A|/2} \leq \Phi(A) \).
Definition. Let $\mathbb{F}_q$ be a finite field, then the nonzero elements $\mathbb{F}_q^*$ are a group and have a character $\chi_q : \mathbb{F}_q^* \rightarrow \mathbb{T}$ such that

$$\chi_q(x) = \begin{cases} 1 & \text{if } x = y^2 \text{ for } y \in \mathbb{F}_q^* \\ 0 & \text{otherwise} \end{cases}.$$ 

This is called the \textbf{quadratic character}.

Exercise 5. Prove that this is indeed a character, i.e., $\chi(xy) = \chi(x)\chi(y)$. (Hint: count.)

Definition. Let $\psi : (\mathbb{F}_q, +) \rightarrow \mathbb{T}$ be an additive character and let $\chi : \mathbb{F}_q^* \rightarrow \mathbb{T}$ be a multiplicative character. Extend $\chi$ to $\mathbb{F}_q$ by setting $\chi(0) = 0$. Define the \textbf{Gaussian sum} to be $S(\chi, \psi) = \sum_{g \in \mathbb{F}_q} \chi(g)\psi(g)$.

Exercise 6. In the above setting prove that if $\psi \neq \psi_0$ and $\chi \neq \chi_0$, then $|S(\chi, \psi)| = \sqrt{q}$.

Exercise 7. Let $G = (\mathbb{F}_q, +)$ where $q$ is a power of an odd prime and let $Q$ be the set of squares of $\mathbb{F}_q$, so $Q = \{x^2 \mid x \in \mathbb{F}_q, x \neq 0\}$. We know that $|Q| = (q - 1)/2$. Prove that $\Phi(Q) \leq \sqrt{q}$. 