Exercise 1 (Clearing the Corner.). A game is played on a grid in the first quadrant of the plane. Each square in the grid can contain at most one coin. The only legal move is to pick a coin such that there is no coin in the square $N$ immediately above it, and no coin in the square $E$ immediately to the right of it. When such a coin is picked, it is removed, and a coin is placed on each of the squares $E$ and $N$ (see Figure 1).

![Figure 1. Here, we can only pick the circled coin.](image)

Starting with a single coin in the bottom left corner, prove that there is no sequence of moves which will empty the boxed region of coins.

Exercise 2. Spreading Infection. Define two squares of the $n \times n$ checkerboard to be neighbors if they share an edge. Each square can be “infected” or not. On each turn, the infection spreads to those squares that have at least two neighbors that are infected (see Figure 2). Prove that at least $n$ initial infections are necessary for the infection to spread to every square on the checkerboard. - Your answer should be a single word that tells it all. Email this word to the instructor.
Exercise 3. Tiling. A tile is a unit square in the plane with edges parallel to the \( x \) and \( y \)-axes, and with edges labelled by integers. Tiles may be translated, but may not be rotated or flipped. The type of a tile is the ordered 4-tuple of integer labels, starting with the label of the “northernmost” edge and moving clockwise (see Figure 3).

![Figure 3. Examples of tiles and types.](image)

Let \( S \) be a collection of tiles, consisting of only finitely many tile types, and infinitely many tiles of each type that occurs in \( S \). Two tiles may be placed adjacent to one another if the edge they share is labelled with the same number on each tile. Given that the first quadrant of the plane can be tiled using \( S \), prove that the entire plane can be tiled using \( S \).

Exercise 4. Can the \( 8 \times 8 \) chessboard with two diagonally opposite corners removed be tiled with non-overlapping dominoes (\( 1 \times 2 \) rectangles)?

Exercise 5. Find all pairs of squares that can be removed from the \( 8 \times 8 \) chessboard such that the remaining board can be tiled with dominoes.

Exercise 6. Consider 27 unit cubes arranged into a \( 3 \times 3 \times 3 \) cube; label the 27 unit cubes with numbers from 1 to 27. Without loss of generality, assume the center cube is labelled 27. Call two unit cubes adjacent if they share a face. Prove that there is no sequence \( a_1, a_2, \ldots, a_{27} \) such that \( a_{27} = 27 \), \( 1 \leq a_i \leq 27 \) for each \( i \), and the cubes labelled with \( a_i \) and \( a_{i+1} \) are adjacent for each \( i \).

Exercise 7. A single knight is in the corner of a \( 4 \times 4 \) chessboard. Prove that it is impossible for the knight to visit every square in at most 15 chess moves. Find a proof which does not involve case-by-case analysis.
Exercise 8. Prove that the $101 \times 101$ chessboard with one corner removed cannot be tiled with triominos ($1 \times 3$ rectangles).

Exercise 9. Suppose a rectangle $R$ is partitioned into a finite number of smaller rectangles, each of which has edges parallel to the edges of $R$. Suppose as well that each of the smaller rectangles has the property that at least one of its edges is of integer length. Prove that this property also holds for $R$.

Definition (Degree). Let $G$ be a graph. Then the degree of a vertex of $G$ is the number of edges incident to the vertex.

Exercise 10. Suppose every vertex of a graph has degree $\leq a + b + 1$ where $a$ and $b$ are positive integers. Prove that the vertices of the graph can be colored red or blue such that every red vertex has $\leq a$ red neighbors and every blue vertex has $\leq b$ blue neighbors.

Exercise 11. Prove that in the previous problem, the “Lovász toggle” will produce an appropriate coloring of the vertices of the graph in a finite number of steps. The “Lovász toggle” starts from any coloring of the vertices of the graph. If a vertex violates the condition imposed by the problem, select such a vertex and change its color. Repeat.

Definition (Quadratic Residue). An integer $a$ is a quadratic residue modulo $p$ if $a \not\equiv 0 \pmod{p}$ and there exists $x \in \{1, 2, \ldots, p-1\}$ such that $x^2 \equiv a \pmod{p}$. Otherwise $a$ is a quadratic non-residue modulo $p$.

Remark: There are exactly $\frac{p-1}{2}$ quadratic residues and $\frac{p-1}{2}$ quadratic non-residues modulo $p$.

Definition (Legendre symbol). The Legendre symbol is defined as

$$\left( \frac{a}{p} \right) := \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } a \text{ is a quadratic residue } \pmod{p} \\ -1 & \text{if } a \text{ is a quadratic non-residue } \pmod{p} \end{cases}$$

Theorem. For integers $a, b$ and prime $p$,

$$\left( \frac{ab}{p} \right) = \left( \frac{a}{p} \right) \left( \frac{b}{p} \right).$$

Definition (Multiplicative character of $\mathbb{F}_p$). Let $\mathbb{F}_p$ denote the field $\mathbb{Z}/p\mathbb{Z}$. Then $\chi : \mathbb{F}_p \to \mathbb{C}$ is a multiplicative character of $\mathbb{F}_p$ if $\chi(0) = 0$, $\chi(1) = 1$, and $\chi(a)\chi(b) = \chi(ab)$ for all $a, b \in \mathbb{F}_p$.

Remark: If $\chi$ is a multiplicative character of $\mathbb{F}_p$, then $\chi(a)$ is a $(p-1)^{st}$ root of unity for $a \neq 0$.

Definition (Primitive root of unity). $z$ is a primitive $n^{th}$ root of unity if $z^n = 1$ but $z^k \neq 1$ for $0 < k < n$.

Definition (Order). The order of a number $z$ is the smallest $n$ such that $z^n = 1$. The order of $z$ is denoted $ord(z)$.
Remark: In \( \mathbb{C} \), the number of primitive \( n^{th} \) roots of unity is \( \varphi(n) \) where \( \varphi \) is Euler’s totient function.

**Theorem.** Let \( n \) be a positive integer. Then
\[
n = \sum_{d|n} \varphi(d).
\]

**Exercise 12.** Let \( a, b \in \mathbb{F}_p \). If \( \text{ord}(a) \) and \( \text{ord}(b) \) are relatively prime, prove that \( \text{ord}(ab) = \text{ord}(a)\text{ord}(b) \).

**Exercise 13.** Let \( a, b \in \mathbb{F}_p \). Prove that
\[
\frac{\text{lcm}(\text{ord}(a), \text{ord}(b))}{\text{gcd}(\text{ord}(a), \text{ord}(b))} \mid \text{ord}(ab) \mid \text{lcm}(\text{ord}(a), \text{ord}(b))
\]

**Exercise 14.** *Invasion of the South.* Consider a grid on the plane, with grid lines parallel to the axes, and grid lines spaced one unit apart. Two grid squares are declared to be adjacent if they share a common vertex or edge. The region \( y > 0 \) is “the North,” and \( y < 0 \) is “the South.” Each grid square can contain at most one coin.

A legal move consists of selecting two adjacent squares \( A \) and \( B \) which each have a coin. If \( A \) has center \((x_1, y_1)\) and \( B \) has center \((x_2, y_2)\), then let \( C \) be the square with center \((2x_2 - x_1, 2y_2 - y_1)\). If \( C \) already has a coin in it, do nothing. If \( C \) has no coin in it, then remove the coins from \( A \) and \( B \), and place a coin in \( C \).

Suppose coins are initially placed only in the North. Prove that no matter how these coins are initially placed, there is no sequence of legal moves that will result in a coin being placed in a square with center \((x, y)\) where \( y < -100 \). Improve \(-100\) to a tight bound.

![Figure 4. Invasion of the South.](image-url)