Definition (Riemann zeta function). The zeta function is defined as
\[ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \]
This converges for \( s > 1 \).
We can also write this as
\[ \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left( \frac{1}{1 - \frac{1}{p^s}} \right) \]
by factoring \( n^s \) into primes and using the geometric series formula.

Exercise 1. Let
\[ p_n = \Pr_{x,y \in [n]} (\gcd(x,y) = 1) \]
where \([n] = \{1,2,\ldots,n\}\) and \( x \) and \( y \) are chosen independently and uniformly at random. Assume that \( \lim_{n \to \infty} p_n \) exists. Show that it must be \( 1/\zeta(2) \). (This should be a two-line proof.)

Exercise 2. What is
\[ \sum_{i=1}^{n} \varphi(i)/n, \]
as a function of \( n \), for large \( n \)? (The formula will be very simple.)

Theorem. If \( p \) is an odd prime, then the Legendre symbol of \( -1 \) is
\[ \left( \frac{-1}{p} \right) = \begin{cases} 1, & p \equiv 1 \pmod{4} \\ -1, & p \equiv -1 \pmod{4} \end{cases} \]

Definition. A Paley tournament is a tournament on \( p \) vertices, numbered 0, 1, \ldots, \( p - 1 \), where \( i \) beats \( j \) iff \( i - j \) is a quadratic residue mod \( p \).

Note that the theorem above implies that, when \( p \equiv -1 \pmod{4} \), a Paley tournament really is a tournament.

Theorem. For \( p > ck^24^k \), a Paley tournament of size \( p \) is \( k \)-paradoxical.

Definition. A multiplicative character \( \chi \) is a function from the integers mod \( p \) to \( \mathbb{C} \) such that \( \chi(0) = 0, \chi(1) = 1, \) and \( \chi(ab) = \chi(a)\chi(b) \). The order of \( \chi \) is the least \( d \) such that
\[ \chi(x)^d = 1 \]
for all nonzero $x$.

Example: let $\chi(a) = \left(\frac{a}{p}\right)$. The order of this character is 2, since it takes only the values $-1$ and $1$ on nonzero $a$.

**Exercise 3.** Prove that if $\chi$ is not the principal character (defined as $\chi_0(a) = 1$ for all $a \not\equiv 0 \pmod{p}$), then

$$\sum_{a=0}^{p-1} \chi(a) = 0.$$

**Theorem** (André Weyl’s character sum estimate). Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree $d$. Assume that $f$ cannot be written as $f(x) = cg(x)^d$ for any constant $c$ and polynomial $g(x)$. Then

$$\left| \sum_{x=0}^{p-1} \chi(f(x)) \right| \leq (k-1)\sqrt{p}.$$

**Definition.** The **Stirling number of the second kind** $S(n,k)$ is the number of partitions of $[n]$ into $k$ nonempty parts.

Last time we saw the Bell number $B_n$, which is the total number of partitions of $[n]$: it is clear from the definitions that

$$B_n = \sum_{k=0}^{n} S(n,k).$$

**Exercise 4.** Prove that

$$\sum_{k=0}^{n} S(n,k) \cdot x(x-1)(x-2) \cdots (x-k+1) = x^n.$$

**Exercise 5.** Prove that

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

**Exercise 6.** Let $p(x) = \sum_n \frac{B_n}{n!} x^n$, the exponential generating function for $B_n$. Use the above recurrence for $B_n$ to show that $p'(x) = e^x p(x)$, and conclude that $p(x) = e^{e^x-1}$.

**Exercise 7.** Show that

$$S(n,k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n,$$

and use this to show that

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$