Definition (Partition function). For a positive integer $n$, the partition function $p(n)$ is the number of ways to write $n = x_1 + \ldots + x_k$ for some $k \in \mathbb{N}$ and some $x_1 \geq x_2 \geq \ldots \geq x_k > 0$.

Theorem. (Hardy-Ramanujan) (stated) 

$$p(n) \sim \frac{1}{4\sqrt{3n}}e^{\sqrt{\frac{2n}{3}}\pi}.$$ 

Theorem. (proved using generating functions) For all $n$, $p(n) < e^{\sqrt{\frac{2n}{3}}\pi}$. 

Exercise 1. Let $F_n$ denote the $n^{th}$ Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Let $F(x) = \sum_{n=1}^{\infty} F_n x^n$. Find a simple closed form expression for $F(x)$.

Exercise 2. Let $a_1, \ldots, a_k$ and $d_1, \ldots, d_k$ be positive integers. Prove that 

$$\frac{1}{1-x} = \frac{x^{a_1}}{1-x^{d_1}} + \ldots + \frac{x^{a_k}}{1-x^{d_k}}$$

implies that $d_i = d_j$ for some $i \neq j$.

Exercise 3. Let $B_n$ denote the Bell numbers, i.e., the number of partitions of a set of $n$ elements. Let $B(x) = \sum_{n=1}^{\infty} \frac{B_n}{n!} x^n$. Prove that $B(x) = e^{e^x-1}$. 

Hint: find a recurrence for $B_n$.

Exercise 4. Prove that there is a constant $b$ such that if a graph on $n$ vertices is universal for all graphs on $k$ vertices, then $n > b^k$. 

Hint: The solution should be no more than two or three lines. (This is a partial converse to a result we proved earlier: there is a constant $c$ such that if $n > c^{k^2}2^k$, then there exists a graph on $n$ vertices that is universal for all graphs on $k$ vertices.)

Theorem. Let $U$ be the unit distance graph in the plane. Let $\chi(U)$ denote the chromatic number of $U$. Then $4 \leq \chi(U) \leq 7$.

Let $U_k(n)$ denote the maximum number of unit distances we can have in a collection of $n$ points in $\mathbb{R}^k$.

Exercise 5. Prove that $U_3(n) = \Theta(n^2)$.

Exercise 6. Does there exist $\epsilon > 0$ such that $U_3(n) = O(n^{2-\epsilon})$?
Exercise 7. Let $G$ be the unit distance graph on $n$ points in the plane. Then $G$ does not contain a $K_{2,3}$, i.e., a complete bipartite graph on two and three vertices.

Exercise 8. If $G$ is a graph with $n$ vertices that does not contain a $K_{2,3}$, then the number of edges of $G$ is $O(n^{3/2})$. 