BABAI DISCRETE MATHEMATICS REU 2008
EXERCISES FROM LECTURE 9, FRI JULY 18

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The cardgame "SET:" 81 cards corresponding to $\mathbb{F}_3^4$; a set of 3 cards x, y, z is called a "SET" if $x + y + z = 0$; the task is to find a "SET" among 12 cards laid out on the table.

Definition. An *affine line* in F^n is a translate of a 1-dimensional subspace, i. e., a subset of the form $a + U$ where U is a one-dimensional subspace.

Note that is $|F| = q$ then each affine line has q elements.

Exercise 1. Prove: three distinct points $x, y, z \in \mathbb{F}_3^n$ form an affine line exactly if $x + y + z = 0$.

Definition (Independent set in $\mathbb{F}_3^n$)**.** A subset $S \subseteq \mathbb{F}_3^n$ is **independent** if it does not contain an affine line.

Exercise 2. Let α_n = the maximum size of an independent subset in $\mathbb{F}_3^n$. Show that $\alpha_3 = 9$.

Definition (Super-multiplicative)**.** A function $f : \mathbb{N} \to \mathbb{R}^+$ is **super-multiplicative** if $f(n + m) \geq f(n)f(m)$ for all $m, n \in \mathbb{N}$.

Exercise 3. Prove that if f is super-multiplicative then $\lim_{n\to\infty} f(n)^{1/n}$ exists and equals $\sup_n f(n)^{1/n}$.

Exercise 4. Prove: the sequence α_n is supermultiplicative.

Therefore the limit $L := \lim_{n\to\infty} \alpha_n^{1/n}$ exists.

Exercise 5. Prove: (a) $2 \leq L \leq 3$. (b) $2.07 \leq L$.
(c)*** (OPEN PROBLEM) Is $L < 3$?

*****Exercise 6.** Prove that $\lim_{n\to\infty} \frac{\alpha_n}{3^n} = 0$.

Exercise 7. Let $S_k(n) = \sum_{t\geq 0} \binom{n}{kt}$. Prove that $|S_3(n) - \frac{2^n}{3}| < 1$.

Exercise 8. Give a closed-form expression for $S_k(n)$.