REU'09 · Transfinite Combinatorics · Lecture 1

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Note: All references of the form [07: A.B.C] refer to item A.B.C from Lecture A of the notes from 2007, available at
http://people.cs.uchicago.edu/∼laci/REU07.

As a warmup and invitation, consider exercises [07: 1.1.1, 1.1.2, 1.1.3]

1.1 Graphs

We covered k-colorable graphs and the chromatic number of a graph. [07: 1.1, p.2] A *bipartite graph* is a 2-colorable graph, for example the graph of the possible moves of a knight on a chessboard. The complete graph with n vertices K_n has vertex set $\{1, \ldots, n\}$ and $\binom{n}{2}$ edges, one between every pair of distinct vertices.

Exercise 1.1.1. Construct a graph G of chromatic number 4 without K_3 as a subgraph (we then say that G is *triangle-free*), having 11 vertices and 5-fold symmetry.

Exercise 1.1.2. Show that for all k there exists a triangle-free graph G with $\chi(G) \geq k$.

Exercises [07: 1.1.5 – 1.1.7]

1.2 Cardinal and Ordinal Numbers

See definitions [07: 1.2.1, 1.2.2, 1.3.1, 1.4.1 – 1.4.4, 1.4.6, 1.4.8, 1.4.10 – 1.4.12, 1.5.1, 1.5.3] for this material. Exercises [07: 1.2.3, 1.4.5 – 1.4.7].

Exercise 1.2.1. If X is a subset of the real numbers $\mathbb{R}$ and X is well-ordered, then $|X| \leq \aleph_0$.

Proposition 1.2.2 (Transfinite Induction)**.** *Let X be a well-ordered set and $Y \subseteq X$ a subset. Suppose that:*

$$\forall x \in X, \textit{ if } (\forall z \in X)(z < x \Longrightarrow z \in Y) \textit{ then } x \in Y.$$

Then $Y = X$.

Proof. Suppose to the contrary that $Y \neq X$. Let $\overline{Y} = X \setminus Y$. Since X is well-ordered and $\overline{Y}$ is nonempty, there exists a first element $x \in \overline{Y}$. Now if $z < x$ then $z \in Y$ so by our assumption, $x \in Y$, which is a contradiction. □

Exercise [07: 1.3.4], after recalling the axiom of choice [07: 1.3.3]. We covered the well-ordering theorem [07: 1.5.4]. Exercises [07: 2.1.2, 2.1.6].

Some more graph-theoretic exercises:

Exercise 1.2.3. If G is an infinite graph, then either G or $\overline{G}$, the complement of G, contains an infinite clique. (A clique is a complete subgraph.)

Exercise 1.2.4. Construct a graph G with vertex set $V = \mathbb{R}$ such that neither G nor $\overline{G}$ contain an uncountable clique. If $|V| > |R| = \mathfrak{c}$, then no such graph exists.