REU'09 · Transfinite Combinatorics · Lecture 10

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10.1 Graph coloring

Proposition 10.1.1. *Let $G = (V, E)$ be a graph such that every vertex has degree $\leq k$. Then $\chi(G) \leq k + 1$.*

The example $G = K_k$ shows that this bound is tight.

Proof. We need to exhibit a coloring of G by $k+1$ colors. By Erdős-deBruijn, we may assume that G is finite. Color the graph inductively as follows: order the vertices $1, \dots, n$ and assume that vertices $1, \dots, j-1$ have been colored. Color vertex j according to the rule:

$$\text{color}(j) = \min\{t \mid (\forall i < j)(\text{if } i \sim j \text{ then color}(i) \neq t\ )\}$$

(Here $\sim$ indicates adjacency.) Since j is connected to at most k other vertices, we only need $k + 1$ choices of colors to employ this rule. □

The algorithm defined in the above proof is called the **greedy coloring algorithm**, because it makes its choices without foresight. Note that the outcome of greedy coloring depends on the ordering of the vertices.

Exercise 10.1.2. Prove: if a graph is k-colorable, there is an ordering of the vertices so that the greedy coloring algorithm will not use more than k colors.

Exercise 10.1.3 (Greedy coloring can be very poor)**.** For every n, construct a bipartite graph with n vertices such that the greedy coloring algorithm uses a *lot* of colors ($> Cn$ for some constant C).

Exercise 10.1.4. If G has no triangles and has n vertices, then $\chi(G) \leq 2\sqrt{n} + 1$.

Exercise 10.1.5. Suppose that the set of vertices V forms an ordered set under $<$ and that every vertex has at most k neighbors to the left. Prove that $\chi(G) \leq 2k + 1$.

This will follow from:

Exercise 10.1.6. Suppose that G is a directed graph such that each vertex has out-degree $\leq k$. Then $\chi(G) \leq 2k + 1$.

Definition 10.1.7. G is planar if there exists a drawing of G on the plane with no intersection of edges.

Theorem 10.1.8. *K_5 and $K_{3,3}$ are not planar. Furthermore, G is not planar if and only if G contains a topological version of K_5 or $K_{3,3}$ as a subgraph (***Kuratowski's Theorem***).*

Here a *topological version* of a graph G means a graph G' whose underlying topological space is homeomorphic to G (subdivide some of the edges by new vertices of degree 2).

Exercise 10.1.9. Every finite planar graph has a vertex of degree ≤ 5. (Hint. Use Euler's formula about the number of vertices, edges, and regions.)

Proposition 10.1.10. *Every planar graph is 6-colorable.*

Proof. We can assume that $G = (V, E)$ is finite, and then induct on the number of vertices. By the previous exercise, there exists some vertex v of degree ≤ 5. By the inductive hypothesis, the subgraph G' of G with vertices $V \setminus \{v\}$ is 6-colorable. Since there are at most 5 edges connecting v with G', we may safely color v with one of the 6 colors. □

Notice that the above proof used the greedy coloring algorithm, but with a *smarter* ordering of the vertices, based on the fact that the property "has a vertex of degree ≤ 5" is *hereditary*, meaning that it is inherited by any subgraph.

With an eye toward proving Exercise 10.1.6, consider:

Exercise 10.1.11. Suppose that each vertex of a directed graph has out-degree $\leq k$. Then G has a vertex of total degree $\leq 2k$.

Since the statement "each vertex has out-degree $\leq k$" is a hereditary property of digraphs, Exercise 10.1.11 will provide an ordering to perform the greedy coloring algorithm and prove Exercise 10.1.6.

Definition 10.1.12. The *girth* of a graph is the length of the shortest cycle. The *odd-girth* is the length of the shortest cycle of odd length.

Notice that a tree has infinite girth and a bipartite graph has infinite odd-girth. Recall that for all k there exists a triangle-free graph G with $\chi(G) \geq k$. As a generalization, we have:

Theorem 10.1.13 (Erdős)**.** *For all finite k, g, there exists a graph G with $\chi(G) \geq k$ and $\text{girth}(G) \geq g$.*

There is a generalization to very large chromatic numbered graphs:

Theorem 10.1.14. *For any infinite cardinal $\mathfrak{m}$ and finite g, there exists a graph G with $\chi(G) \geq \mathfrak{m}$ and $\text{odd-girth}(G) \geq g$.*

However, we cannot get this result for general girth, only for odd-girth, because of the following:

Theorem 10.1.15 (Erdős - Hajnal)**.** *If $\chi(G) \geq \aleph_1$, then G contains a 4-cycle ($K_{2,2}$). In fact, for ℓ finite, G contains $K_{\ell,\aleph_1}$.*