10.1 Graph coloring

Proposition 10.1.1. Let $G = (V, E)$ be a graph such that every vertex has degree $\leq k$. Then $\chi(G) \leq k + 1$.

The example $G = K_k$ shows that this bound is tight.

Proof. We need to exhibit a coloring of $G$ by $k+1$ colors. By Erdős-deBruijn, we may assume that $G$ is finite. Color the graph inductively as follows: order the vertices $1, \ldots, n$ and assume that vertices $1, \ldots, j-1$ have been colored. Color vertex $j$ according to the rule:

$$\text{color}(j) = \min\{ t \mid (\forall i < j)(i \sim j \text{ then color}(i) \neq t) \}$$

(Here $\sim$ indicates adjacency.) Since $j$ is connected to at most $k$ other vertices, we only need $k + 1$ choices of colors to employ this rule.

The algorithm defined in the above proof is called the **greedy coloring algorithm**, because it makes its choices without foresight. Note that the outcome of greedy coloring depends on the ordering of the vertices.

Exercise 10.1.2. Prove: if a graph is $k$-colorable, there is an ordering of the vertices so that the greedy coloring algorithm will not use more than $k$ colors.

Exercise 10.1.3 (Greedy coloring can be very poor). For every $n$, construct a bipartite graph with $n$ vertices such that the greedy coloring algorithm uses a *lot* of colors ($> Cn$ for some constant $C$).
Exercise 10.1.4. If $G$ has no triangles and has $n$ vertices, then $\chi(G) \leq 2\sqrt{n} + 1$.

Exercise 10.1.5. Suppose that the set of vertices $V$ forms an ordered set under $<$ and that every vertex has at most $k$ neighbors to the left. Prove that $\chi(G) \leq 2k + 1$.

This will follow from:

Exercise 10.1.6. Suppose that $G$ is a directed graph such that each vertex has out-degree $\leq k$. Then $\chi(G) \leq 2k + 1$.

Definition 10.1.7. $G$ is planar if there exists a drawing of $G$ on the plane with no intersection of edges.

Theorem 10.1.8. $K_5$ and $K_{3,3}$ are not planar. Furthermore, $G$ is not planar if and only if $G$ contains a topological version of $K_5$ or $K_{3,3}$ as a subgraph (Kuratowski’s Theorem).

Here a topological version of a graph $G$ means a graph $G'$ whose underlying topological space is homeomorphic to $G$ (subdivide some of the edges by new vertices of degree 2).

Exercise 10.1.9. Every finite planar graph has a vertex of degree $\leq 5$. (Hint. Use Euler’s formula about the number of vertices, edges, and regions.)

Proposition 10.1.10. Every planar graph is 6-colorable.

Proof. We can assume that $G = (V, E)$ is finite, and then induct on the number of vertices. By the previous exercise, there exists some vertex $v$ of degree $\leq 5$. By the inductive hypothesis, the subgraph $G'$ of $G$ with vertices $V \setminus \{v\}$ is 6-colorable. Since there are at most 5 edges connecting $v$ with $G'$, we may safely color $v$ with one of the 6 colors. \hfill \Box

Notice that the above proof used the greedy coloring algorithm, but with a smarter ordering of the vertices, based on the fact that the property “has a vertex of degree $\leq 5$” is hereditary, meaning that it is inherited by any subgraph.

With an eye toward proving Exercise 10.1.6, consider:

Exercise 10.1.11. Suppose that each vertex of a directed graph has out-degree $\leq k$. Then $G$ has a vertex of total degree $\leq 2k$. 


Since the statement “each vertex has out-degree \( \leq k \)” is a hereditary property of digraphs, Exercise 10.1.11 will provide an ordering to perform the greedy coloring algorithm and prove Exercise 10.1.6.

**Definition 10.1.12.** The *girth* of a graph is the length of the shortest cycle. The *odd-girth* is the length of the shortest cycle of odd length.

Notice that a tree has infinite girth and a bipartite graph has infinite odd-girth. Recall that for all \( k \) there exists a triangle-free graph \( G \) with \( \chi(G) \geq k \). As a generalization, we have:

**Theorem 10.1.13** (Erdős). *For all finite \( k, g \), there exists a graph \( G \) with \( \chi(G) \geq k \) and girth\( (G) \geq g \).*

There is a generalization to very large chromatic numbered graphs:

**Theorem 10.1.14.** *For any infinite cardinal \( m \) and finite \( g \), there exists a graph \( G \) with \( \chi(G) \geq m \) and odd-girth\( (G) \geq g \).*

However, we cannot get this result for general girth, only for odd-girth, because of the following:

**Theorem 10.1.15** (Erdős - Hajnal). *If \( \chi(G) \geq \aleph_1 \), then \( G \) contains a 4-cycle \( (K_{2,2}) \). In fact, for \( \ell \) finite, \( G \) contains \( K_{\ell,\aleph_1} \).*