REU'09 · Transfinite Combinatorics · Lecture 13

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13.1 Regressive Functions

Recall:

Definition 13.1.1. A function $g : \omega_1 \to \omega_1$ is **regressive** if for all $\alpha \geq 1$, $g(\alpha) < \alpha$.

Theorem 13.1.2 (Fodor's Theorem, baby version)**.** *If $g : \omega_1 \to \omega_1$ is regressive, then there is $\beta \in \omega_1$ such that $|g^{-1}(\beta)| = \aleph_1$.*

Proof. Since ω_1 is well-ordered, if we choose some point and keep applying g we will always reach 0 after a finite number of steps. Let A_k be the set of α such that $g^{k-1}(\alpha) > 0, g^k(\alpha) = 0$. We can show by induction that if the theorem is false, each A_k is countable. But their union is ω_1. This is a contradiction. □

Question 13.1.3. Is there a "large" subset $S \subset \omega_1$ such that there is a regressive $g : S \to \omega_1$, so that the preimage of each point is countable?

If S is the set of successor ordinals, we can simply "subtract 1." This means that Fodor's theorem holds even if we restrict the domain of g to limit ordinals! In fact, we can make S even bigger; for example, limit ordinals with an "immediate limit ordinal predecessor"; i.e. those of the form $\omega(\beta+1)$, or even $\omega^n(\beta+1)$ (which we can map to $\omega^n\beta$).

Question 13.1.4. Let $P = \{\omega^\alpha : \alpha < \omega_1\}$. Can we define a regressive function on the complement of P without any point with $\aleph_1$ preimages?

Yes! Let $g(\alpha) = \sup\{\gamma < \alpha, \gamma \in P\}$. We could easily make this even sparser, replacing ω with, say ϵ_0.

Definition 13.1.5. $S \subseteq \omega_1$ is **stationary** if for every closed, cofinal set $C \subseteq \omega_1$, $S \cap C \neq \emptyset$.

Exercise 13.1.6. Show that every stationary set is cofinal.

Note 13.1.7. If S is *not* stationary, then there is $g : S \to \omega_1$ regressive such that for all $\beta \in \omega_1$, $|g^{-1}(\beta)| \leq \aleph_0$ because there is a closed, cofinal set C disjoint from S and we can define g as above (with C in the role of P).

Exercise 13.1.8. If S is stationary, then for every regressive $g : S \to \omega_1$ there is $\beta \in \omega_1$ such that $|g^{-1}(\beta)| = \aleph_1$.

Theorem 13.1.9 (Fodor's theorem)**.** *If $S \subset \omega_1$ is stationary and $g : S \to \omega_1$ is regressive then there is $\beta \in \omega_1$ such that $g^{-1}(\beta)$ is stationary.*

13.2 Measurable Cardinals

σ-additive or "countably additive" means "less-than-$\aleph_1$-additive." We shall drop the "less than" and call it $\aleph_1$-additive.

Definition 13.2.1. Let κ be an infinite cardinal. A $(0, 1)$ measure μ on a set A is **κ-additive** if the union of fewer than κ-many 0-sets is still a 0-set. If μ is $\aleph_0$-additive we say it's **finitely-additive**, and if it's $\aleph_1$-additive we say it's **σ-additive**.

Note that this means that if μ is κ-additive and $|B| < \kappa$ then $\mu(B) = 0$.

Exercise 13.2.2. Suppose there exists an $\aleph_1$-additive measure. Let $\mathfrak{m}$ be the smallest cardinal for which such a μ exists. Show that μ is $\mathfrak{m}$-additive.

Definition 13.2.3. $\mathfrak{m}$ is a **measurable** cardinal if there is an $\mathfrak{m}$-additive measure on $\mathfrak{m}$.

Exercise 13.2.4. If $\mathfrak{m}$ is measurable, then $\mathfrak{m}$ is strongly inaccessible. That is:

(a) $cf(m) = m$ (cofinalty), i. e., $\mathfrak{m}$ cannot be written as the sum of fewer smaller cardinals; and

(b) if $\mathfrak{n} < \mathfrak{m}$ then $2^{\mathfrak{n}} < \mathfrak{m}$.

Proof. $cf(\mathfrak{m}) = \mathfrak{m}$: suppose $\mathfrak{m}$ is measurable by μ, and $cf(\mathfrak{m}) < \mathfrak{m}$. So $\mathfrak{m} = \sup_{\alpha \in I} \mathfrak{n}_\alpha$ where $|I| < \mathfrak{m}$. Now $\mu(\mathfrak{n}_\alpha) = 0$ for each α, but there are fewer than $\mathfrak{m}$ of them and their union is all of $\mathfrak{m}$, a contradiction. □

Theorem 13.2.5. *If $\mathfrak{m}$ is a measurable cardinal, and S is the set of all cardinals $\mathfrak{n} < \mathfrak{m}$ that are strongly inaccessible, then $|S| = \mathfrak{m}$; in fact, S is stationary in $\mathfrak{m}$.*

Exercise 13.2.6. Prove : if $\mathfrak{m}$ is measurable then $\mathfrak{m} \to (\mathfrak{m}, \mathfrak{m})$.

Exercise 13.2.7. Prove: if $\mathfrak{m} \to (\mathfrak{m}, \mathfrak{m})$, then $\mathfrak{m}$ is strongly inaccessible.