13.1 Regressive Functions

Recall:

Definition 13.1.1. A function $g : \omega_1 \to \omega_1$ is **regressive** if for all $\alpha \geq 1$, $g(\alpha) < \alpha$.

Theorem 13.1.2 (Fodor’s Theorem, baby version). If $g : \omega_1 \to \omega_1$ is regressive, then there is $\beta \in \omega_1$ such that $|g^{-1}(\beta)| = \aleph_1$.

Proof. Since $\omega_1$ is well-ordered, if we choose some point and keep applying $g$ we will always reach 0 after a finite number of steps. Let $A_k$ be the set of $\alpha$ such that $g^{k-1}(\alpha) > 0, g^k(\alpha) = 0$. We can show by induction that if the theorem is false, each $A_k$ is countable. But their union is $\omega_1$. This is a contradiction. \qed

Question 13.1.3. Is there a “large” subset $S \subset \omega_1$ such that there is a regressive $g : S \to \omega_1$, so that the preimage of each point is countable?

If $S$ is the set of successor ordinals, we can simply “subtract 1.” This means that Fodor’s theorem holds even if we restrict the domain of $g$ to limit ordinals! In fact, we can make $S$ even bigger; for example, limit ordinals with an “immediate limit ordinal predecessor”; i.e. those of the form $\omega(\beta + 1)$, or even $\omega^n(\beta + 1)$ (which we can map to $\omega^n\beta$).

Question 13.1.4. Let $P = \{\omega^\alpha : \alpha < \omega_1\}$. Can we define a regressive function on the complement of $P$ without any point with $\aleph_1$ preimages?
Yes! Let $g(\alpha) = \sup\{\gamma < \alpha, \gamma \in P\}$. We could easily make this even sparser, replacing $\omega$ with, say $\epsilon_0$.

**Definition 13.1.5.** $S \subseteq \omega_1$ is **stationary** if for every closed, cofinal set $C \subseteq \omega_1$, $S \cap C \neq \emptyset$.

**Exercise 13.1.6.** Show that every stationary set is cofinal.

**Note 13.1.7.** If $S$ is not stationary, then there is $g : S \to \omega_1$ regressive such that for all $\beta \in \omega_1$, $|g^{-1}(\beta)| \leq \aleph_0$ because there is a closed, cofinal set $C$ disjoint from $S$ and we can define $g$ as above (with $C$ in the role of $P$).

**Exercise 13.1.8.** If $S$ is stationary, then for every regressive $g : S \to \omega_1$ there is $\beta \in \omega_1$ such that $|g^{-1}(\beta)| = \aleph_1$.

**Theorem 13.1.9** (Fodor’s theorem). If $S \subseteq \omega_1$ is stationary and $g : S \to \omega_1$ is regressive then there is $\beta \in \omega_1$ such that $g^{-1}(\beta)$ is stationary.

### 13.2 Measurable Cardinals

$\sigma$-additive or “countably additive” means “less-than-$\aleph_1$-additive.” We shall drop the “less than” and call it $\aleph_1$-additive.

**Definition 13.2.1.** Let $\kappa$ be an infinite cardinal. A $(0, 1)$ measure $\mu$ on a set $A$ is **$\kappa$-additive** if the union of fewer than $\kappa$-many 0-sets is still a 0-set. If $\mu$ is $\aleph_0$-additive we say it’s **finitely-additive**, and if it’s $\aleph_1$-additive we say it’s **$\sigma$-additive**.

Note that this means that if $\mu$ is $\kappa$-additive and $|B| < \kappa$ then $\mu(B) = 0$.

**Exercise 13.2.2.** Suppose there exists an $\aleph_1$-additive measure. Let $m$ be the smallest cardinal for which such a $\mu$ exists. Show that $\mu$ is $m$-additive.

**Definition 13.2.3.** $m$ is a **measurable** cardinal if there is an $m$-additive measure on $m$.

**Exercise 13.2.4.** If $m$ is measurable, then $m$ is strongly inaccessible. That is:

(a) $\text{cf}(m) = m$ (cofinality), i.e., $m$ cannot be written as the sum of fewer smaller cardinals; and
(b) if $n < m$ then $2^n < m$.

Proof. $cf(m) = m$: suppose $m$ is measurable by $\mu$, and $cf(m) < m$. So $m = \sup_{\alpha \in I} n_\alpha$ where $|I| < m$. Now $\mu(n_\alpha) = 0$ for each $\alpha$, but there are fewer than $m$ of them and their union is all of $m$, a contradiction. \qed

**Theorem 13.2.5.** If $m$ is a measurable cardinal, and $S$ is the set of all cardinals $n < m$ that are strongly inaccessible, then $|S| = m$; in fact, $S$ is stationary in $m$.

**Exercise 13.2.6.** Prove: if $m$ is measurable then $m \rightarrow (m, m)$.

**Exercise 13.2.7.** Prove: if $m \rightarrow (m, m)$, then $m$ is strongly inaccessible.