REU'09 · Transfinite Combinatorics · Lecture 14

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14.1 Measurable Cardinals

Theorem 14.1.1. *If κ is the smallest cardinal with a σ-additive $(0,1)$-measure μ, then μ is κ-additive.*

Proof. Suppose there is some $\lambda < \kappa$ such that there are λ many 0-sets $\{A_\alpha : \alpha \in \lambda\}$ whose union has measure 1. We can define a σ-additive measure on λ: if $S \subset \lambda$, let $\mu^*(S) = \mu(\cup_{\alpha\in S} A_\alpha)$. It is easy to check that this is a σ-additve measure on λ. □

Theorem 14.1.2. *If κ is measurable, then $\kappa \to (\kappa, \kappa)$.*

Proof. Color a vertex red if the set of neighbours connected by red edges has measure 1, and blue if the set of blue neighbours does. Assume w.l.o.g. that the set of red vertices has measure 1. We claim that there is a red clique of size κ.

The red edges form a graph where every vertex is adjacent to almost every vertex. To find a clique of size κ, we proceed by transfinite recursion: suppose v_β for $\beta < \alpha$ has been chosen. Choose v_α as the first vertex that is adjacent to all v_β for $\beta < \alpha$; such a vertex exists, because we've picked fewer than κ so far. □

Theorem 14.1.3. *If $\kappa \to (\kappa, \kappa)$, then κ is strongly inaccessible.*

Proof. (a) $cf(\kappa) = \kappa$: assume, for a contradiction, that $cf(\kappa) = \lambda < \kappa$, so κ is a union of λ many sets $\{A_\alpha : \alpha < \lambda\}$, each smaller than κ; we may assume that they are disjoint. Color the edges within each A_α red, and edges between any two of the sets blue. This is a coloring of the complete graph with neither a red clique nor a blue clique of size κ.

(b) $\lambda < \kappa$ implies $2^\kappa < \kappa$: assume, for a contradiction, that $\lambda \leq \kappa \leq 2^\lambda$. We'll show that $2^\lambda \not\to (\lambda^+, \lambda^+)$; clearly, this implies $\kappa \not\to (\kappa, \kappa)$. Hint: Genaralize Sierpinski's coloring which we constructed to show $2^{\aleph_0} \not\to (\aleph_1, \aleph_1)$. Recall the construction: consider two orderings of $2^{\aleph_0}$: the natural ordering of the reals, and a well-ordering. Now put a red edge between a and b if the two orderings agree on them; and a blue edge if they disagree. Any red clique is a well-ordered subset of the reals so it is countable; and any blue clique is a reverse-well-ordered subset of the reals, so again it is countable.

□

Exercise 14.1.4. Complete part (b) of the proof of Theorem 14.1.3 by generalizing Sierpinski's argument to showing $2^\lambda \not\to (\lambda^+, \lambda^+)$.

14.2 Uncountable chromatic number

Theorem 14.2.1 (Erdős-Hajnal)**.** *If $\chi(G) \geq \aleph_1$, then $G \supset K_{2,2}$; in fact, $G \supset K_{\ell,\aleph_1}$ for all $\ell < \omega$.*

In class, we proved this theorem for the case when G has $\aleph_1$ vertices. The proof was a slight variation on the proof given on Day 7 two years ago.

It started with this exercise.

Exercise 14.2.2. If $G = H_1 \cup H_2$, then $\chi(G) \leq \chi(H_1)\chi(H_2)$.

Exercise 14.2.3. Extend the proof of the Erdős-Hajnal theorem to the genaral case where the number of vertices is $\aleph_\gamma$ by transfinite induction on γ.