REU'09 · Transfinite Combinatorics · Lecture 2

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July 15, 2009

2.1 First-order logic

In first-order logic, one encounters the following types of objects:

Variables: Examples: $x_1, x_2, x_3, \ldots, y_1, \ldots$

Operation Symbols: Examples:

Binary: $+$, $\cdot$

Unary: $x \mapsto x^{-1}$

Nullary: 0, 1 (nullary operations are called "constants")

Relation Symbols: Example of a binary relation symbol: $<$

Terms: Objects built up from variables using operation symbols, a.k.a. "polynomials." Example: $(x_1 + x_2)x_3 + x_1x_4$

Predicates: Relations between terms. Example: If t_1 and t_2 are terms, an **atomic formula** is a statement of the form $t_1 = t_2$ or $t_1 < t_2$, where $<$ is a relation symbol. If R is a ternary relation symbol then $R(t_1, t_2, t_3)$ is an atomic formula.

Operation symbols can have any arity ≥ 0 and relation symbols can have any arity ≥ 1.

We can build up more general formulas from atomic formulas using **logical connectives** and **quanitifiers**. The four logical connectives are $\wedge$ (AND), $\vee$ (OR), $\neg$ (NOT), and $\rightarrow$ (IMPLIES). Thus, if ϕ_1 and ϕ_2 are formulas, then we can form the formulas $\phi_1 \wedge \phi_2$, $\phi_1 \wedge \phi_2$, $\neg\phi_1$, and $\phi_1 \rightarrow \phi_2$.

There are two quantifiers, the **universal quantifier** $\forall$ and the **existential quantifier** $\exists$. If ϕ is a formula, x a variable, we can form formulas $(\forall x)(\phi)$ and $(\exists x)(\phi)$.

A *formula* is a string of symbols built up from atomic formulas by repeated application of the above rules. (This is an inductive definition.)

For every formula ϕ, we have a set $F(\phi)$ of "free variables" of the formula. We define this notion inductively. If ϕ is an atomic formula, then

$$F(\phi) = \{\text{all variables occurring in } \phi\}.$$

We extend this to all formulas inductively by the following rules:

$$\begin{aligned}
F(\phi_1 * \phi_2) &= F(\phi_1) \cup F(\phi_2), \text{ where } * \in \{\vee, \wedge, \rightarrow\} \\
F(\neg\phi) &= F(\phi), \\
F((\forall x)(\phi)) &= F(\phi) \setminus \{x\}, \\
F((\exists x)(\phi)) &= F(\phi) \setminus \{x\}.
\end{aligned}$$

Definition 2.1.1. A **sentence** is a formula without free variables.

Definition 2.1.2. A **language** is a list of of operation and relation symbols (each of a given arity). An **interpretation** of a language is a set A equipped with a map $A^k \to A$ for every k-ary operation in the language, and a subset of A^k (or equivalently a map $A^k \to \{0, 1\}$) for every k-ary relation.

Given an interpretation of the language and an assignment of values from A to each variable, each formula receives a truth value (definable by induction on the length of the formula). (Quantifiers range over A.)

Discussion 2.1.3. Consider the formula

$$(x = y) \vee (\forall x)(\exists y)(x = y^2).$$

"Is this statement true or false in the complex numbers?" It is neither true nor false. To get a definite truth values, we have to plug in for the free variables. If $x = 5$, $y = 5$, then it is true. If $x = 5$, $y = 6$, then it is also true. But over the reals, it false if we do not plug in the same value for x and y, even in the case $x = 25$, $y = 5$.

Exercise 2.1.4. Define free occurrences of variables.

Note 2.1.5. Under all interpretations and substitutions,

$$\text{“}\phi_1 \to \phi_2\text{” is true} \iff \text{“}\neg\phi_1 \vee \phi_2\text{” is true.}$$

2.2 Models of first-order sentences

The language of a **poset** consists of no operations, and a single binary relation, $<$. It has axioms like

$$(\forall x, y, z)((x < y) \wedge (y < z) \rightarrow (x < z)).$$

The language of graphs consists of no operations, and a single binary relation, which we call **adjaceny**, $x \sim y$. We have the following axioms.

Symmetry:

$$(\forall x, y)(x \sim y \rightarrow y \sim x) \tag{2.2.1}$$

Irreflexivity:

$$(\forall x)(\neg x \sim x) \tag{2.2.2}$$

Question 2.2.1. What graph properties can be expressed by first-order sentences?

Connectivity *cannot* be expressed by a first-order sentence. "Every vertex has degree 3" *can* be expressed by a first-order sentence:

$$(\forall x)(\exists y_1, y_2, y_3)(x \sim y_1 \wedge x \sim y_2 \wedge x \sim y_3 \wedge y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3 \wedge \\ (\forall z)((z \neq y_1 \wedge z \neq y_2 \wedge z \neq y_3) \rightarrow z \not\sim x)). \tag{2.2.3}$$

Definition 2.2.2. A **model** of a first-order sentence is a structure on which the first-order sentence is true.

Example 2.2.3. Three-regular graphs are a model for the first-order sentence that is the $\wedge$ of the sentences (2.2.1), (2.2.2), and (2.2.3).

Definition 2.2.4. A **field** is a structure with binary operations $+$ and $\cdot$ and nullary operations 0 and 1 that satisfies the following axioms.

(a) $0 \neq 1$

(b) $(\forall x, y, z)((x + y) + z = x + (y + z))$

(c) $(\forall x, y)(x + y = y + x)$

(d) $(\forall x)(x + 0 = x)$

(e) $(\forall x)(\exists y)(x + y = 0)$

(f) $(\forall x, y, z)((x \cdot y) \cdot z = x \cdot (y \cdot z))$

(g) $(\forall x, y)(x \cdot y = y \cdot x)$

(h) $(\forall x)(x \cdot 1 = x)$

(i) $(\forall x)(\exists y)(x \neq 0 \to x \cdot y = 1)$

(j) $(\forall x, y, z)(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$

Definition 2.2.5. The **charactristic** of a field is p if $\underbrace{1 + \cdots + 1}_{p} = 0$, and p is the smallest positive such number. (It is known that if such p exists, it is a prime.) If there is no such p, we say the field has characteristic 0 (or ∞); if p does exist, we say the field has "finite charateristic."

The property of characteristic 0 can be axiomatized by an infinite list of axioms, but the property of finite characteristic cannot. Characteristic 0 is axiomatized by

$$\begin{aligned} 1 + 1 &\neq 0, \\ 1 + 1 + 1 &\neq 0, \\ 1 + 1 + 1 + 1 + 1 &\neq 0, \\ &\vdots \end{aligned}$$

Exercise 2.2.6. Let $\mathcal{M}$ be a class of structures of the same language. If both $\mathcal{M}$ and its complement $\overline{\mathcal{M}}$ are axiomatizable, then they are *finitely* axiomatizable.

Exercise 2.2.7. Are connected or disconnected graphs axiomatizable among finite graphs?

Exercise 2.2.8. We generate a random graph on a given set of n vertices by flipping a coin for each pair of vertices to decide adjacency. If ϕ is a sentence in teh language of graphs then for every n, let $p_n(\phi) = \Pr(\text{random graph on } n \text{ vertices satisfies } \phi)$. Prove **Fagin's theorem**: For every sentence ϕ, $\lim_{n\to\infty} p_n(\phi) = 0$ or 1.

Exercise 2.2.9. A finite field has order q, a power of a prime. Construct a first-order sentence in the language of fields such that the *finite* models of the sentence are exactly the fields of order $q \equiv 1 \pmod 4$. (We don't care what the infinite models will be.)

Definition 2.2.10. Let $\mathcal{S}$ be a set of sentences. Assume that $\mathcal{S}$ has an infinite model. We say that $\mathcal{S}$ is **categorical** in cardinality $\mathfrak{m}$ if $\mathcal{S}$ has exactly one model (up to isomorphism) of cardinality $\mathfrak{m}$.

Note 2.2.11. Some sets of sentences have finite but not infinite models. For example, the sentence $(\exists x)(\forall y)(y = x)$ has only the one-element set as a model.

Exercise 2.2.12. If $\mathcal{S}$ has infinitely many finite models, then $\mathcal{S}$ has an infinite model.

Definition 2.2.13. A the axioms of a **dense ordering** are the same as those for a linear ordering, along with

$$(\forall x, y)(x < y \to (\exists z)(x < z < y)),$$

$$(\forall x)(\exists y)(y < x),$$
$$(\forall x)(\exists y)(y > x).$$

Example 2.2.14. The rational numbers and the real numbers are densely ordered.

Exercise 2.2.15. (1) All countable dense linear orders are order-isomorphic to $(\mathbb{Q}, <)$. In other words, the axioms of dense linear order are categorical in the countable cardinality. (2) It is not categorical in any uncountable cardinality.

Exercise 2.2.16. Find $\mathcal{S}$ such that for every uncountable cardinality $\mathfrak{m}$, there exists a unique model of cardinality $\mathfrak{m}$, but for $\mathfrak{m} = \aleph_0$ there are infinitely many nonisomorphic models. In other words, $\mathcal{S}$ is categorical in all uncountable cardinalities but not in $\aleph_0$. (Hint: a class of fundamental structures in abstract algebra.)