REU'09 · Transfinite Combinatorics · Lecture 3

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3.1 Fields and abelian groups

Exercise 3.1.1. Find a sentence satisfied by all finite fields but not by $\mathbb{R}$.

Exercise 3.1.2. Find a sentence satisfied by all finite fields but not by $\mathbb{C}$.

Exercise 3.1.3. Find a sentence ϕ such that the finite fields that satisfy ϕ are precisely those of order $\equiv 1 \pmod{m}$.

Question 3.1.4. Is there a sentence that can pick out finite fields of order $\equiv 2 \pmod{7}$? (The instructor does not know the answer.)

Definition 3.1.5. An abelian group $(A, +)$ is **torsion-free** if it has no element of finite order other than 0.

Definition 3.1.6. An abelian group $(A, +)$ is **divisible** if $\forall x \in A$, $\forall n > 0$, $\exists y$ such that $x = ny$.

Example 3.1.7. The additive group of the rational numbers, $(\mathbb{Q}, +)$, are torsion-free and divisible. The group $(\mathbb{Q}/\mathbb{Z}, +)$ is divisible but not torsion-free.

Exercise 3.1.8. If an abelian group is torsion-free and divisible, then it is a vector space over $\mathbb{Q}$. So it is determined up to isomorphism by a single parameter, the dimension.

Both torsion-free abelian groups and divisible abelian groups are defined by first-order theories. So torsion-free divisible abelian groups come from a first-order theory.

Exercise 3.1.9. This theory is not categorical in the countable cardinality but it is categorical in every uncountable cardinality.

The following exercise is needed for the solution.

Exercise 3.1.10. For every cardinal $\mathfrak{m} \geq 1$, if V is an $\mathfrak{m}$-dimensional vector space over $\mathbb{Q}$, then $|V| = \max\{\aleph_0, \mathfrak{m}\}$.

3.2 The Random Graph

Exercise 3.2.1. Consider random graphs on countably many vertices. Then Pr(random graphs X and Y are isomorphic) $= 1$.

We call the unique (up to isomorphism) countable random graph R. This graph was first described by Erdős and Rényi in 1960.

Exercise 3.2.2. R is *universal*: Every countable graph is an induced subgraph of R. (An **induced subgraph** is one in which all edges between included vertices are included in the subgraph.)

Exercise 3.2.3. (a) Any isomorphism between finite subgraphs of R extends to an automorphism of R. (b) $|\mathrm{Aut}\, R| = \mathfrak{c}$.

Theorem 3.2.4 (Morley)**.** *If a first-order theory is categorical in some uncountable cardinal, then it is categorical in all uncountable cardinals.*

Definition 3.2.5. A field F is **algebraically closed** if every non-constant polynomial over F has a root in F.

Exercise 3.2.6. The theory of algebraically closed fields of a given characteristic is not categorical in countable cardinality, but it is categorical in all uncountable cardinalities.

Exercise 3.2.7 (Erdős-de Bruijn)**.** If all finite subgraphs of a graph G are k-colorable (for some $k < \infty$), then G is k-colorable.

Exercise 3.2.8. Use Exercise 3.2.7 to prove that for infinitely many switches, there does not need to be a dictator switch.

3.3 Gödel's compactness theorem

Notation:

- M: model (interpretation of operations and relations)
- $M \models \phi$: ϕ is satsified by M

Definition 3.3.1. A set $\mathcal{S}$ of sentences is **consistent** if it has a model: $(\forall \phi \in \mathcal{S})(M \models \phi)$.

Theorem 3.3.2 (Gödel's compactness theorem)**.** *$\mathcal{S}$ is consistent if and only if every finite subset of $\mathcal{S}$ is consistent.*

Theorem 3.3.3 (Tychonoff's compactness theorem)**.** *The topological product of compact spaces is compact.*

Exercise 3.3.4. Infer Erdős-de Bruijn from

(a) Gödel's compactness theorem;

(b) Tychonoff's compactness theorem;

(c) Zorn's lemma.

Exercise 3.3.5. Prove Zorn's lemma from the well-ordering theorem.

We are looking for finitely additive 0-1 measures over a set A. In other words, we want $\mu : \mathcal{P}(A) \to \{0, 1\}$ such that

(1) $\forall B \subseteq A$, $\mu(B) \in \{0, 1\}$;

(2) $\forall B_1, B_2 \subseteq A$, $B_1 \cap B_2 = \varnothing$, then $\mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2)$;

(3) $\mu(A) = 1$.

Exercise 3.3.6. If $\mu(B_1) = \mu(B_2) = 1$, then $\mu(B_1 \cap B_2) = 1$.

Exercise 3.3.7. If $B_1 \subseteq B_2$ then $\mu(B_1) \leq \mu(B_2)$.

Exercise 3.3.8. If A is finite, then μ has a dictator.

We would thus like to add a fourth condition,

(4) $\forall x \in A$, $\mu(\{x\}) = 0$. (There is no dictator.)

Definition 3.3.9. Let $\mathcal{F} \subseteq \mathcal{P}(A)$. We say that $\mathcal{F}$ has the **finite intersection property** if the intersection of any finite number of sets in $\mathcal{F}$ is not empty.

Theorem 3.3.10. *If $\mathcal{F} \subseteq \mathcal{P}(A)$ has the finite intersection property, then $\exists \mu$ such that $(\forall F \in \mathcal{F})(\mu(F) = 1)$.*

Proof. Let Φ be the set of all subsets of $\mathcal{P}(A)$ with the finite intersection property. Let Φ' be those subsets in Φ that contain $\mathcal{F}$. Φ' is a partially ordered set. Every chain in Φ' is bounded by the union of the members of the chain, as any finitely many elements of the union will be contained in one of the members of the chain, and will thus have nonempty intersection. So by Zorn's lemma, there exists a maximal element $\mathcal{F}_{\max}$ of Φ'. Set

$$\mu(B) = \begin{cases} 1 & \text{if } B \in \mathcal{F}_{\max}, \\ 0 & \text{if } B \notin \mathcal{F}_{\max}. \end{cases}$$

Exercise 3.3.11. Show that μ is OK. (Hint: show that $\forall B \subseteq A$, either $B \in \mathcal{F}_{\max}$ or $\overline{B} \in \mathcal{F}_{\max}$.)

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Exercise 3.3.12. Solve the infinite lamp problem using a finitely additive non-principal (no dictator) 0-1 measure.

Exercise 3.3.13. Prove Erdős-de Bruijn using 0-1 measures.

Exercise 3.3.14. Prove Gödel's compactness theorem using 0-1 measures.

3.4 Ultraproducts

Definition 3.4.1. Let A be a set, μ a non-principal, finitely additive 0-1 measure on a set A, and let be F_i fields for $i \in A$. Let $f, g \in \prod_{i \in A} F_i$. We say that f and g are **almost equal**, and write $f \sim g$, if $\mu(\{i \in A \mid f(i) \neq g(i)\}) = 0$. The **ultraproduct** of the F_i is defined to be $G = \prod_{i \in A} F_i / \sim$.

Exercise 3.4.2. The ultraproduct of fields is a field.

Exercise 3.4.3. Let $A = \omega$. If F_i is finite, $|F_i| \to \infty$, then $|G| = \mathfrak{c}$.

Exercise 3.4.4. If $(\forall i \in A)(|F_i| \equiv 1 \pmod 4)$, then $x^2 + 1 = 0$ is solvable in G.