REU'09 · Transfinite Combinatorics · Lecture 4

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4.1 Cardinals

For finite sets A and B, the number of functions $f : A \to B$ is $|B|^{|A|}$, so for general cardinals, we define

$$|A|^{|B|} = |\{f : A \to B\}|.$$

Addition of cardinals is defined by

$$|A| + |B| = |A \sqcup B|.$$

Multiplication of cardinals is defined by

$$|A| \cdot |B| = |A \times B|.$$

More generally, for a set $\{A_i \mid i \in I\}$ of sets, we define

$$\sum_{i \in I} |A_i| = \left| \bigsqcup A_i \right|,$$

$$\prod_{i \in I} |A_i| = \left| \prod A_i \right|.$$

Exercise 4.1.1. *For cardinals a, b, c, show that*

(a) $a^{b+c} = a^b \cdot a^c$;

(b) $(a^b)^c = a^{b \cdot c}$;

(c) $(a \cdot b)^c = a^c \cdot b^c$.

Exercise 4.1.2. *For all cardinals m, $m < 2^m$, i.e., $m \leq 2^m$ and $m \neq 2^m$.*

Exercise 4.1.3 (Julius König)**.** *If $\{a_i \mid i \in I\}$ and $\{b_i \mid i \in I\}$ are sets of cardinals such that $a_i < b_i$ for all $i \in I$, then $\sum_{i \in I} a_i < \prod_{i \in I} b_i$.*

Note 4.1.4. $\sum_{i \in I} a_i$ is not necessarily less than $\sum_{i \in I} b_i$. Example: let I be countable, $a_i = 1$ and $b_i = 2$ for all i. Then both sums are countable. Similarly, $\prod_{i \in I} a_i$ is not necessarily less than $\prod_{i \in I} b_i$. Example: let again I be countable; let $a_i = 2$ and $b_i = 4$ for all i. Then both products are $2^{\aleph_0}$ (because $4^{\aleph_0} = 2^{2\aleph_0} = 2^{\aleph_0}$).

The cardinality of $[0,1]$ is $\mathfrak{c}$ by definition. We claim that $\mathfrak{c} = 2^{\aleph_0}$. To see this, note that we can take any element of $[0,1]$ and write its binary expansion. However, this does not give a bijection between $[0,1]$ and the set $\{0,1\}^{\aleph_0}$ because, for instance, the expansions

$$.0100111111\ldots \quad \text{and}$$
$$.0101000000\ldots$$

represent the same real number. However, the only numbers with two binary expansions are the dyadic fractions, which are countable. Thus, $\mathfrak{c} + \aleph_0 = 2^{\aleph_0}$. Now let $\mathfrak{d}$ be the cardinality of $[0,1]$ minus the dyadic fractions; so $\mathfrak{c} = \mathfrak{d} + \aleph_0$; therefore $2^{\aleph_0} = \mathfrak{d} + \aleph_0 + \aleph_0 = \mathfrak{d} + \aleph_0 = \mathfrak{c}$.

Exercise 4.1.5. $\mathfrak{c}^2 = \mathfrak{c}$. *(Hint. Use that $\mathfrak{c} = 2^{\aleph_0}$.)*

Next we claim that $|\mathbb{R}| = \mathfrak{c}$. Ideed, as $[0,1] \subset \mathbb{R}$, we have $\mathfrak{c} \leq |\mathbb{R}|$. On the other hand, $\mathbb{R}$ is the union of countably many copies of $[0,1)$, so we get

$$|\mathbb{R}| \leq \mathfrak{c} \cdot \aleph_0 \leq \mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}.$$

Alternately, we can get a bijection between $\mathbb{R}$ and $(0,1)$ using the tangent function.

The number of functions $f : [0,1] \to \mathbb{R}$ is thus $\mathfrak{c}^{\mathfrak{c}} = 2^{\aleph_0 \cdot \mathfrak{c}} = 2^{\mathfrak{c}}$.

Theorem 4.1.6. *Let $C[0,1]$ be the set of continuous functions $f : [0,1] \to \mathbb{R}$. Then $|C[0,1]| = \mathfrak{c}$.*

Proof. Clearly $|C[0,1]| \geq \mathfrak{c}$, as there are $\mathfrak{c}$ many constant functions.

A continuous function is determined by its values on a dense set; in particular, $f \in C[0,1]$ will be determined by $f|_{\mathbb{Q}\cap[0,1]}$. The number of functions $\mathbb{Q} \cap [0,1] \to \mathbb{R}$ is $\mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$, so $|C[0,1]| \leq \mathfrak{c}$. □

Definition 4.1.7. *If $f, g \in C[0,1]$, we define $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. We say f and g are* ***orthogonal*** *and write $f \perp g$ if $\langle f, g\rangle = 0$.*

Exercise 4.1.8. *If $\{f_i \mid i \in I\}$ are orthogonal, $f_i \neq 0$, then $|I| \leq \aleph_0$.*

Exercise 4.1.9. *Prove that there exists a Euclidean space (a vector space over $\mathbb{R}$ with a positive definite inner product) such that*

(1) $|V| = \mathfrak{c}$;

(2) there exist $\mathfrak{c}$ many orthogonal vectors in V.

4.2 Finitely additive 0-1 measures

The material on ultrafilters is covered in section 4.6 of the 2007 notes on Transfinite Combinatorics.

We can rephrase Theorem 3.3.9 as follows.

Theorem 4.2.1. *$\forall \mathcal{F} \subseteq \mathcal{P}(A)$ with the finite intersection property, there exists an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$.*

For A infinite, if we choose $\mathcal{F}$ to be the set of subsets of A that contain all but one element, we obtain the following.

Corollary 4.2.2. *For every infinite set A, there exists a nonprincipal* 0-1 *measure (or equivalently, a nonprincipal ultrafilter).*

This gives a resolution to the infinite switch problem. Suppose the set I of switches has a nonprincipal measure μ. We can write $I = R \sqcup Y \sqcup G$, where R, Y, and G are the set of switches set to red, yellow, and green, respectively. Then $\mu(R) + \mu(Y) + \mu(G) = 1$, so exactly one of R, Y, and G has measure 1 ("almost all switches are in the same position"). The corresponding color will be the lamp color.

To see that this works, suppose that the color is red, and then all of the switches are changed. Let R', Y', and G' be the new sets of red, yellow, and green switches. Then $R' \subseteq Y \cup G$, so R' has measure 0. Thus, the light will change color to either yellow or green.

4.3 Arrow's Paradox

Given k options in an election, $k \geq 3$, each voter will have $k!$ choices as to how to rank the options. We want a function $F : (\underline{k!})^N \to \underline{k!}$ that takes everybody's rankings and gives one ranking for society. We would like a few conditions on this function.

(1) **Unanimity ("Pareto optimality"):** If all voters prefer option A to option B, then the output prefers A to B.

(2) **Independence of irrelevant alternatives:** The ranking of A and B in the outcome depends only on every voter's ranking between A and B.

Theorem 4.3.1 (Arrow, 1963)**.** *Conditions (1) and (2) imply a dictatorship.*

Note. Kenneth Arrow shared the Nobel Prize in economics in 1972.

Exercise 4.3.2. *Arrow's theorem fails if we have infinitely many voters (but still finitely many options).*

4.4 Order types

The material on order-types is covered in sections 1.4 and 1.5 of the 2007 notes on Transfinite Combinatorics. η is the order-type of the rational numbers.

Theorem 4.4.1. *Every countable dense ordering is of type η.*

Proof. Let H be a countable set with a dense ordering. We enumerate the rationals as $r_0, r_1, r_2, \ldots$ and the elements of H as $h_0, h_1, h_2, \ldots$. We define $f : \mathbb{Q} \to H$ as follows. We set $f(r_0) = h_0$. By denseness, there is an element of h below or above h_0, so we let $f(r_1)$ be such an element, depending on whether r_1 is less than or greater than r_0. We continue in this manner to define f completely as a monotone injection of $\mathbb{Q}$ into H. This is not a bijection, howver. Similarly, we can get a monotone injection from H to $\mathbb{Q}$.

Exercise 4.4.2. *Find two order-types α and β such that there exist monotone injections $\alpha \hookrightarrow \beta$ and $\beta \hookrightarrow \alpha$ but no monotone bijection.*

To fix this, we alternate between assigning $f(r_i)$ for the next unassigned i and $f^{-1}(h_j)$ for the next unassigned j. This will give a monotone bijection. □

Exercise 4.4.3. *Another fix: for all i, choose $f(r_i)$ to be the h_j with the smallest j such that the choice is compatible with the ordering of the previous choices. Prove that this results in a bijection.*

Thus, $\eta \cdot 2 = \eta + \eta = \eta$. But $2 \cdot \eta \neq \eta$, because it is not dense.

$2 \cdot \omega = \omega$, but $\omega \cdot 2 = \omega + \omega \neq \omega$ because it has an element that is preceded by infinitely many elements.

Exercise 4.4.4. *An ordered set $(A, <)$ is well-ordered if and only if it does not contain ω^*.*

Definition 4.4.5. *An* ***ordinal*** *is the order-type of a well-ordered set. The* ***standard name*** $\underline{\alpha}$ *of an ordinal α by transfinite recursion as follows: $\underline{\alpha} = \{\underline{\beta} \mid \beta < \alpha\}$.*

Example 4.4.6.

$$
\begin{aligned}
&\underline{0} = \varnothing, \\
&\underline{1} = \{\underline{0}\} = \{\varnothing\}, \\
&\underline{2} = \{\underline{0}, \underline{1}\} = \{\varnothing, \{\varnothing\}\}, \\
&\underline{3} = \{\underline{0}, \underline{1}, \underline{2}\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\
&\vdots \\
&\underline{\omega} = \{\underline{0}, \underline{1}, \underline{2}, \ldots\}, \\
&\vdots
\end{aligned}
$$

Convention: henceforth, "ordinal" will always refer to the standard name of the ordinal; the underline will be omitted.

Exercise 4.4.7. *If α and β are well-ordered, then one is a prefix of the other.*

Exercise 4.4.8. *Every set of ordinals has a supremum (least upper bound).*

We define ordinal exponentiation α^β by transfinite recursion.

Definition 4.4.9. *(a) $\alpha^0 = 1$; (b) $\alpha^{\beta+1} = \alpha^\beta \alpha$; if β is alimit ordinal then $\alpha^\beta = \sup_{\gamma < \beta} \alpha^\gamma$.*

Exercise 4.4.10. *If α and β are countable, then α^β is countable.*

Exercise 4.4.11. *Every ordinal can be uniquely written in the base-ω number system. In other words, any ordinal α can be uniquely written as*

$$\alpha = \omega^{\beta_1} k_1 + \omega^{\beta_2} k_2 + \cdots + \omega^{\beta_s} k_s,$$

where the k_i are finite and the β_i are decreasing.