5.1 Infinite Arrow problem

For the Arrow problem with infinite voters but $k < \infty$ choices, a non-principal finitely additive 0-1 measure on the set of voters will give a non-dictatorial solution. Since everybody must choose between $k!$ options, exactly one of the rankings will be chosen by almost all voters; we choose this ranking as the output. We observe that option $A$ will be ranked higher than option $B$ in the output if and only if $A$ is ranked higher than $B$ by almost all voters. This proves both the unanimity property and the independence of irrelevant alternatives.

5.2 The König path lemma and Erdős-de Bruijn

Definition 5.2.1. A graph is locally finite if every vertex has finite degree.

Definition 5.2.2. A rooted graph is a graph with a choice of a special vertex, called the root.

Definition 5.2.3. A path in a graph is a sequence of edges with no repeated vertices.

Lemma 5.2.4 (König’s path lemma). A locally finite, connected, infinite, rooted graph has an infinite path from the root. For an infinite directed graph, if every vertex is reachable from the root and every vertex has finite out-degree, then there exists an infinite directed path from the root.
Proof. Without loss of generality, we may assume that our graph is acyclic. Since the outdegree of each vertex is finite, and the number of reachable vertices from the root is infinite, one of the vertices attached to the root must have infinitely many vertices reachable from it. We continue inductively to get an infinite path.

We can use the König path lemma to prove the Erdős-de Bruijn theorem for a graph on countably many vertices. Suppose we have a graph $\Gamma$ on $\mathbb{N}$ in which each finite subgraph is $k$-colorable. Consider the graph $C$ whose vertex set is $\bigcup_{n \in \mathbb{N}} C_n$, where $C_n$ is the set of $k$-colorings of the subgraph induced by the vertices $\{1, \ldots, n\}$ of $\Gamma$. We put an edge from $x \in C_n$ to $y \in C_{n+1}$ if the coloring $y$ of $\{1, \ldots, n+1\}$ restricts to the coloring $x$ of $\{1, \ldots, n\}$. The König path lemma then gives us an infinite path from the root in $C$, which defines a $k$-coloring of the graph $\Gamma$.

Exercise 5.2.5. A tile is a square with numbers written on each edge. Two tiles can be put next to each other if the corresponding numbers on the edges are equal. Tiles cannot be rotated. Suppose we have finitely many types of tiles. Show that if it is possible to tile the first quadrant with the given types of tiles, then it is possible to tile the entire plane.

For the proof of Erdős-de Bruijn in full generality using ultrafilters, see Section 6.7 of the 2007 Transfinite Combinatorics notes. Here, we give another proof of Erdős-de Bruijn using nothing more than Zorn’s Lemma.

Proof. Let $G = (V, E)$ be a graph such that all finite subgraphs are $k$-colorable. Consider the set

$$\{E' \supseteq E \mid \text{all finite subgraphs of } (V, E') \text{ are } k\text{-colorable}\}.$$ 

If $E \subseteq E_1 \subseteq E_2 \subseteq \cdots$ with every finite subgraph of $(V, E_i)$ $k$-colorable, then any finite subgraph of $(V, \bigcup_i E_i)$ will be contained in some $(V, E_j)$ by finiteness, and hence will be $k$-colorable. Thus, by Zorn’s lemma, there is a supergraph $(V, E^*)$ of $G$ maximal with respect to the property that every finite subgraph is $k$-colorable. For finite graphs, such a graph must be a complete $k$-partite graph, i.e., we can split the vertices into $k$ classes such that each vertex is connected to every vertex in every class other than its own. Equivalently, the complement of the graph is the union of $k$ cliques, i.e., nonadjacency is an equivalence relation.
Exercise 5.2.6. Prove that non-adjacency in \((V, E^*)\) is an equivalence relation.

\[\square\]

5.3 Ultraproducts

**Definition 5.3.1.** Let \(A\) be a set, \(\mu\) a 0-1 measure on \(A\), and \(F_i\) fields, \(i \in A\). Let \(f, g \in \prod_{i \in A} F_i\). We say that \(f\) and \(g\) are **almost equal**, and write \(f \sim g\), if \(\mu(\{i \in A \mid f(i) \neq g(i)\}) = 0\). The **ultraproduct** of the \(F_i\) is defined to be \(G = \prod_{i \in A} F_i / \sim\).

**Claim 5.3.2.** If \(f \sim f'\), and \(g \sim g'\), then \(f + g \sim f' + g'\).

**Proof.** For almost all \(\alpha\), \(f(\alpha) = f'(\alpha)\), and for almost all \(\alpha\), \(g(\alpha) = g'(\alpha)\). So for almost all \(\alpha\), both of these hold. Thus, for almost all \(\alpha\), \(f(\alpha) + g(\alpha) = f'(\alpha) + g'(\alpha)\). \(\square\)

**Exercise 5.3.3.** \([f] + [g] + [h] = [f] + ([g] + [h])\).

All of the standard ring axioms hold for the ultraproduct. We claim that the ultraproduct is in fact a field.

We define \(\frac{1}{[f]} = [g], [f] \neq [0]\), by

\[
g(\alpha) = \begin{cases} 
\frac{1}{f(\alpha)} & \text{if } f(\alpha) \neq 0, \\
* & \text{if } f(\alpha) = 0.
\end{cases}
\]

Since \(f(\alpha) \neq 0\) for almost all \(\alpha\), what we assign for the \(\alpha\) where \(f(\alpha) = 0\) is irrelevant.

We can likewise define the ultraproducts of graphs and other structures.

**Theorem 5.3.4 (Loś).** Let \(\phi(x_1, \ldots, x_n)\) be a first-order formula, \(\{A_\alpha \mid \alpha \in I\}\). Suppose \(\mu\) is a finitely additive 0-1 measure on \(I\). We set \(B = \prod A_\alpha / \mu\). Then for \([b_1], \ldots, [b_n] \in B\), \(B \models \phi([b_1], \ldots, [b_n])\) if and only for almost every \(\alpha\), \(A_\alpha \models \phi(b_1(\alpha), \ldots, b_n(\alpha))\).

**Exercise 5.3.5.** No ultraproduct of finite fields is isomorphic to \(\mathbb{R}\).

**Example 5.3.6.** What is the countable ultrapower of the infinite path? The infinite path satisfies the sentences “every edge has degree 2” and, for every \(n \geq 3\) “there are no cycles of length \(n\).” The graphs that satisfy these conditions are disjoint unions of infinite paths. So the countable ultrapower must be a disjoint union of \(\aleph\) many infinite paths.
Example 5.3.7. What is the ultraproduct of all finite cycles? Every finite cycle satisfies “every edge has degree 2,” and for every $n$, almost every cycle satisfies “there are no cycles of length $n$.” So this ultraproduct must also be a disjoint union of $c$ many infinite paths.

Example 5.3.8. If we replace the finite cycles above with the disjoint union of two finite cycles, we again get the same ultraproduct. This implies that connectedness of finite graphs cannot be expressed by a first-order sentence (i.e., there is no first order sentence $\phi$ such that the finite models of $\phi$ are exactly the connected finite graphs, regardless of what the infinite models of $\phi$ may be).