REU'09 · Transfinite Combinatorics · Lecture 6

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6 First order definability

Theorem 6.0.1. *If S_i is a collection of finite sets and $|S_i| \to \infty$, then $|\prod_i S_i/\mu| = 2^{\aleph_0}$, the cardinality of the continuum.*

Theorem 6.0.2 (Łoś)**.** *If $\phi(x_1, \ldots, x_n)$ is a formula with $x_1, \ldots, x_n$ free variables, then for all $\underline{a}_1, \ldots, \underline{a}_n \in \mathcal{B}$,*

$$\mathcal{B} \vDash \phi(\underline{a}_1, \ldots, \underline{a}_n) \iff \text{for a.a. } i, \quad \mathcal{A}_i \vDash \phi(a_1[i], \ldots, a_n[i]).$$

Theorem 6.0.3. *Connectedness of graphs is not an elementary property; that is, it cannot be described by a (possibly even infinite) set of first-order sentences.*

Proof. Let $\mathcal{B} = \prod \mathcal{A}_i/\mu$. If we have a sentence ϕ, then $\mathcal{B} \vDash \phi$ if and only if for almost all i, $\mathcal{A}_i \vDash \phi$, by Łoś' theorem.

Now, suppose that connectedness is an elementary property. That would mean that an ultraproduct of connected graphs is connected. So we just need to find connected graphs whose ultraproduct is disconnected.

For the graphs, take infinite paths. The following statements hold of an infinite path, and so must hold in the ultraproduct:

(a) every vertex has degree 2.

(b) there are no cycles of length n, for any n. (This is infinitely many sentences.)

So the ultraproduct has continuum-many vertices, every one has degree 2, and there are no cycles. So the ultraproduct must be continuum many disjoint infinite paths. (In fact, this theory is categorical in all uncountable cardinalities.) □

Theorem 6.0.4. *An ultraproduct of disconnected graphs is disconnected.*

Proof. In each factor of the product, take two elements, in different connected components. Make formulas $\phi_n(x, y)$ saying "there is no path of length n between x and y." Now apply Łoś' theorem. □

This means that the same proof won't work to show that disconnectedness is not an elementary property. However, we can use elementary equivalence:

Definition 6.0.5. $\mathcal{A}$ and $\mathcal{B}$ are **elementary equivalent** if for all sentences ϕ,

$$\mathcal{A} \vDash \phi \iff \mathcal{B} \vDash \phi.$$

We can now show that disconnectedness is not an elementary property: a single infinite path is elementary equivalent to continuum many, since the latter is an ultrapower of the former. (It's an easy corollary of Łoś' theorem that an ultrapower of a structure is elementary equivalent to that structure.)

We can write sentences all of whose *finite* models are connected graphs: "if there are n vertices, they are connected." But there's no single sentence (or, equivalently, finite set of sentences) that works.

Theorem 6.0.6. *There exist connected finite graphs $\mathcal{A}_i$ and disconnected finite connected graphs $\mathcal{B}_i$ such that*

$$\prod \mathcal{A}_i/\mu \cong \prod \mathcal{B}_i/\mu.$$

Proof. As the $\mathcal{A}_i$, take cycles of increasing length; for the $\mathcal{B}_i$, pairs of cycles. In both cases, the ultraproduct is continuum many lines (check the axioms above, and apply the fact that they're categorical in uncountable cardinalities). □

Exercise 6.0.7. Prove Łoś' theorem by formula induction.

Theorem 6.0.8 (Gödel's Compactness Theorem)**.** *A set of formulas is consistent if and only if every finite subset is consistent.*

Proof. We can prove this using ultrapducts. If $\mathcal{S}$ is a set of sentences and $F \subseteq \mathcal{S}$ is a finite subset, there is some model $\mathcal{A}_F \vDash F$; take an ultraproduct of these sets, as we did in the proof of Erdős-deBruijn. □

Theorem 6.0.9 (Erdős-deBruijn)**.** *If every finite subgraph of a graph is k-colorable, then the graph itself is k-colorable.*

Exercise 6.0.10. Deduce the Erdős - deBruijn Theorem from Gödel's Compactness Theorem.

Hint: Introduce a constant (nullary operation) for each node. Represent the coloring by k unary relations. For each edge, state that its endpoints don't have the same color.

Definition 6.0.11. If a, b, and c are cardinalities, we write

$$a \to (b, c)$$

to mean that in every graph G of cardinality a, either there is a clique of cardinality b or $\overline{G}$ has a clique of cardinality c.

Show:

(a) $\aleph_0 \to (\aleph_0, \aleph_0)$.

(b) $2^{\aleph_0} \not\to (\aleph_1, \aleph_1)$; in other words, there's a graph G whose cardinality is the continuum, but neither G nor $\overline{G}$ contain an uncountable clique.

(c) $(2^{\aleph_0})^+ \to (\aleph_1, \aleph_1)$.