1 Vector Spaces

1.1 Basics

Definition 1.1.1. A vector space (over the reals) is a set $V$ with

(a) addition $V \times V \rightarrow V$, $(x, y) \mapsto x + y$, and

(b) scalar multiplication $\mathbb{R} \times V \rightarrow V$, $(\alpha, x) \mapsto \alpha x$,

satisfying following axioms

(a) $(V, +)$ is an abelian group, i.e.,

(a1) $(\forall x, y \in V)(\exists! x + y \in V)$,

(a2) $(\forall x, y \in V)(x + y = y + x)$ (commutative law),

(a3) $(\forall x, y, z \in V)((x + y) + z = x + (y + z))$ (associative law),

(a4) $(\exists 0 \in V)(\forall x)(x + 0 = 0 + x = x)$ (existence of zero),

(a5) $(\forall x \in V)(\exists(-x) \in V)(x + (-x) = 0)$,

b) (b1) $(\forall x \in V)(\forall \alpha \in \mathbb{R})(\exists! \alpha x \in V)$

(b2) $(\forall \alpha, \beta \in \mathbb{R})(\forall x \in V)(\alpha(\beta x)) = (\alpha\beta)x$ (“associativity” linking two operations),

(b3) $(\forall \alpha, \beta \in \mathbb{R})(\forall x \in V)((\alpha + \beta)x = \alpha x + \beta x)$ (distributivity over scalar addition),

(b4) $(\forall \alpha \in \mathbb{R})(\forall x, y \in V)(\alpha(x + y) = \alpha x + \alpha y)$ (distributivity over vector addition),
(c) \((\forall x \in V)(1 \cdot x = x)\) (normalization).

**Exercise 1.1.2.** Show \((\forall x \in V)(0x = 0)\). (The first 0 is a number, the second a vector.)

**Exercise 1.1.3.** Show \((\forall \alpha \in \mathbb{R})(\alpha 0 = 0)\).

**Exercise 1.1.4.** Show \((\forall \alpha \in \mathbb{R})(\forall x \in V)(\alpha x = 0 \iff (\alpha = 0 \text{ or } x = 0))\)

**Example 1.1.5.** Examples of vector spaces

1. \(\mathbb{R}^n\) - the set of \(n\)-tuples of real numbers
2. geometric vectors in 2 or 3 dimensions
3. \(k \times \ell\) matrices
4. \(C[0,1]\), the continuous functions from \([0,1]\) to \(\mathbb{R}\)
5. the space of infinite sequences of real numbers
6. \(\mathbb{R}[x] = \{\text{polynomials with real coefficients}\}\).
7. \(\mathbb{R}(x) = \{\text{rational functions with real coefficients}\}\) (see Definition 1.1.6 below).
8. \(\mathbb{R}^\Omega\), the functions from \(\Omega\) to \(\mathbb{R}\)
   - \(\Omega = \{1,\ldots,n\}\) is example 1 above
   - \(\Omega = \{1,\ldots,k\} \times \{1,\ldots,\ell\}\) is example 3
   - \(\Omega = [0,1]\) contains example 4
   - \(\Omega = \mathbb{N}\) is example 5

**Definition 1.1.6.** Rational functions are equivalence classes of formal fractions \(\frac{p}{q}\) with \(p, q \in \mathbb{R}[x], q \neq 0\), under the equivalence relation \(\frac{p_1}{q_1} = \frac{p_2}{q_2}\) iff \(p_1q_2 = p_2q_1\).

**Definition 1.1.7.** A linear combination of vectors \(v_1,\ldots,v_k \in V\) is a vector \(\alpha_1v_1 + \cdots + \alpha_kv_k\) where \(\alpha_1,\ldots,\alpha_k \in \mathbb{R}\).

**Definition 1.1.8.** The span of \(v_1,\ldots,v_k \in V\) is the set of all linear combinations of \(v_1,\ldots,v_k\), i.e., \(\text{Span}(v_1,\ldots,v_k) = \{\alpha_1v_1 + \cdots + \alpha_kv_k | \alpha_1,\ldots,\alpha_k \in \mathbb{R}\}\).

**Remark 1.1.9.** We let \(\text{Span}(\emptyset) = \{0\}\), by virtue of the fact that an empty sum is always zero.

**Definition 1.1.10.** A linear combination of an infinite list of vectors is a linear combination of a finite sublist. The span of an infinite list \(S\) of vectors is, as defined before, as the set of all linear combinations of \(S\).
Note that $0 \in \text{Span}(T)$ for any list $T$ of vectors, because the trivial linear combination (all coefficients are 0) is 0.

**Exercise 1.1.11.** Prove: $(\forall S \subseteq V)(\text{Span}(\text{Span}(S)) = \text{Span}(S))$.

**Definition 1.1.12.** A list $S$ of vectors generates $V$ if $\text{Span}(S) = V$.

**Definition 1.1.13.** $U \subseteq V$ is a subspace if $U = \text{Span}(U)$. Notation: $U \leq V$.

**Exercise 1.1.14.** Prove: a subset $U \subseteq V$ is a subspace exactly if it is closed under linear combinations.

**Exercise 1.1.15.** Prove: a subset $U \subseteq V$ is a subspace exactly if

(a) $0 \in U$;

(b) $(\forall x, y \in U)(x + y \in U)$;

(c) $(\forall x \in U)(\forall \alpha \in \mathbb{R})(\alpha x \in U)$.

**Exercise 1.1.16.** Prove: a subspace is a vector space (under the same operations).

**Exercise 1.1.17.** Show $\forall S \subseteq V, \text{Span}(S) \leq V$, i.e. that $\text{Span}(S)$ is a subspace of $V$. (Recall that $\text{Span}(\emptyset) = \{0\}$.)

**Exercise 1.1.18.** Show that $\text{Span}(S)$ is the smallest subspace containing $S$, i.e.,

(a) $S \subseteq \text{Span}(S) \leq V$ and

(b) If $S \subseteq W \leq V$ then $\text{Span}(S) \leq W$.

**Exercise 1.1.19.** Show that the intersection of any set of subspaces is a subspace.

**Remark 1.1.20.** Note that this doesn’t hold true for unions.

**Exercise 1.1.21.** If $U_1, U_2 \leq V$ then $U_1 \cup U_2$ is a subspace if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

**Exercise 1.1.22.** $\text{Span}(S) = \bigcap_{S \subseteq W \leq V} W$.

### 1.2 Linear Independence

**Definition 1.2.1.** Vectors $v_1, \ldots, v_k \in V$ are linearly independent if only their trivial linear combination gives 0, i.e., $(\forall \alpha_1, \ldots, \alpha_k \in \mathbb{R})(\alpha_1 v_1 + \cdots + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \cdots = \alpha_k = 0.)$

**Definition 1.2.2.** An infinite list of vectors is linearly independent if all finite sublists are linearly independent.

**Exercise 1.2.3.** Prove: The empty list is linearly independent.
Exercise 1.2.4. Prove: A one-element lists \{v\} of vectors is linearly independent ⇐⇒ v ≠ 0.

Exercise 1.2.5. A list containing 0 is never linearly independent.

Exercise 1.2.6. A list of vectors with repetitions is never linearly independent.

Exercise 1.2.7. Show that if \( T \subseteq S \subseteq V \) and \( S \) is linearly independent then \( T \) is linearly independent.

Exercise 1.2.8. Find a curve in \( \mathbb{R}^n \), i.e., a continuous function \( f : \mathbb{R} \rightarrow \mathbb{R}^n \), such that any \( n \) points on the curve are linearly independent, i.e., for any \( n \) distinct real numbers \( \alpha_1, \ldots, \alpha_n \), the vectors \( f(\alpha_1), \ldots, f(\alpha_n) \) are linearly independent. (Give a simple explicit formula.)

Definition 1.2.9. We say that vectors \( u, v \in V \) are parallel if \( u, v \neq 0 \) and \( (\exists \alpha \in \mathbb{R})(u = \alpha v) \).

Exercise 1.2.10. Show that vectors \( u, v \in V \) are linearly dependent if and only if \( u = 0 \) or \( v = 0 \) or \( u, v \) are parallel.

Remark 1.2.11. We say that a property \( P \) is a finitary property if a set \( S \) has the property \( P \) if and only if all finite subsets of \( S \) have property \( P \). Note that linear independence is, by Definition 1.2.2, a finitary property.

Exercise 1.2.12. \(^{\star}\) (Erdős – deBruijn) Show that 3-colorability of a graph is a finitary property. (The same holds for 4-colorability, etc.) (HINTS for three different proofs. (1) Use Zorn’s Lemma. (2) Use the Compactness Theorem of first-order logic. (3) Use Tikhonoff’s Theorem (compactness of the product of compact topological spaces).

Exercise 1.2.13. Let \( \alpha_1, \ldots, \alpha_n \) be distinct real numbers. Prove: \( \frac{1}{x-\alpha_1}, \ldots, \frac{1}{x-\alpha_n} \) are linearly independent rational functions.

Exercise 1.2.14. Prove: for all \( \alpha, \beta \in \mathbb{R}, \sin(x), \sin(x+\alpha), \sin(x+\beta) \) are linearly dependent functions \( \mathbb{R} \rightarrow \mathbb{R} \).

Exercise 1.2.15. Prove: \( 1, \cos(x), \cos(2x), \cos(3x), \ldots, \sin(x), \sin(2x), \ldots \) are linearly independent functions \( \mathbb{R} \rightarrow \mathbb{R} \).

1.3 Rank, Dimension

Definition 1.3.1. A maximal linearly independent subset of a set \( S \subseteq V \) is a subset \( T \subseteq S \) such that

(a) \( T \) is linearly independent, and

(b) if \( T \subseteq T' \subseteq S \) then \( T' \) is linearly dependent.

Definition 1.3.2. A maximum linearly independent subset of a set \( S \subseteq V \) is a subset \( T \subseteq S \) such that
(a) $T$ is linearly independent, and

(b) if $T' \subseteq S$ is linearly independent then $|T| \geq |T'|$.

**Exercise 1.3.3.** (Independence of vertices in a graph.) Find a graph which has a maximal independent set of vertices which is not maximum.

We shall see that this cannot happen with linear independence: every maximal linearly independent set is maximum.

**Exercise 1.3.4.** Let $S \subseteq V$. Then there exists $T \subseteq S$ such that $T$ is a maximal independent subset of $S$.

**Exercise 1.3.5.** Let $L \subseteq S \subseteq V$. Assume $L$ is linearly independent. Then there exists a maximal linearly independent subset $T \subseteq S$ such that $L \subseteq T$. (Every linearly independent subset of $S$ set can be extended to a maximal linearly independent subset of $S$.)

**Remark 1.3.6.** It is easy to prove Exx. 1.3.4, 1.3.5 by successively adding vectors until our set becomes maximal as long as all linearly independent subsets of $S$ are finite. For the infinite case, we need “transfinite induction” or a result from set theory called Zorn’s Lemma (a version of the Axiom of Choice).

**Definition 1.3.7.** A vector $v \in V$ depends on $S \subseteq V$ if $v \in \text{Span}(S)$, i.e. $v$ is a linear combination of $S$.

**Definition 1.3.8.** A set of vectors $T \subseteq V$ depends on $S \subseteq V$ if $T \subseteq \text{Span}(S)$.

**Exercise 1.3.9.** Show that dependence is transitive: if $R \subseteq \text{Span}(T)$ and $T \subseteq \text{Span}(S)$ then $R \subseteq \text{Span}(S)$. (“A linear combination of linear combinations is a linear combination.”)

**Definition 1.3.10.** A vector $v$ depends on a list $S$ of vectors if $v \in \text{Span}(S)$. A list $T$ of vectors depends on a list $S$ of vectors if every member of $T$ depends on $S$.

**Exercise 1.3.11.** Prove: dependence is transitive, i.e., if $R, S, T$ are lists of vectors, $R$ depends on $S$ and $S$ depends on $T$ then $R$ depends on $T$ (“linear combinations of linear combinations are linear combinations”).

**Exercise 1.3.12.** Suppose that $\sum \alpha_i v_i$ is a nontrivial linear combination. Then ($\exists i$) such that $v_i$ depends on the rest (i.e. on $\{v_j \mid j \neq i\}$). Indeed, this will be the case whenever $\alpha_i \neq 0$.

**Exercise 1.3.13.** A list $S$ of vectors is linearly dependent iff some member of the list depends on the rest.

**Exercise 1.3.14.** If $v_1, \ldots, v_k$ are linearly independent and $v_1, \ldots, v_k, v_{k+1}$ are linearly dependent then $v_{k+1}$ depends on $v_1, \ldots, v_k$.

The next fundamental result asserts the impossibility of boosting linear independence.
Theorem 1.3.15 (First Miracle of Linear Algebra). If $v_1, \ldots, v_k$ are linearly independent and they all depend on $w_1, \ldots, w_\ell$ then $k \leq \ell$.

We shall prove this in Section 1.6. Now we derive some corollaries.

Corollary 1.3.16. All maximal linearly independent subsets of a set $S \subseteq V$ are maximum.

Exercise 1.3.17. If $T \subseteq S$, $T$ is a maximal independent subset of $S$ then $S \subseteq \text{Span}(T)$.

Exercise 1.3.18. Prove Corollary 1.3.16 from Theorem 1.3.15 and Exercise 1.3.17.

Definition 1.3.19. For $S \subseteq V$, the rank of $S$ is the common cardinality of all the maximal independent subsets of $S$. Notation: $\text{rk}(S)$.

Definition 1.3.20. The dimension of a vector space is $\dim(V) := \text{rk}(V)$.

Exercise 1.3.21. $\text{rk}(S) = \text{rk}(\text{Span}(S)) = \dim(\text{Span}(S))$.

Exercise 1.3.22. Show that $\dim(\mathbb{R}^n) = n$.

Exercise 1.3.23. Let $P_k$ be the space of polynomials of degree $\leq k$. Show that $\dim(P_k) = k+1$.

Exercise 1.3.24. Let $T = \{\sin(x + \alpha) | \alpha \in \mathbb{R}\}$. Prove $\text{rk}(T) = 2$.

Notation: $[n] = \{1, 2, \ldots, n\}$.

Exercise 1.3.25. Let us say that the set system $A_1, \ldots, A_m \subseteq [n]$ is $k$-intersecting if $(\forall i \neq j)(|A_i \cap A_j| = k)$. Show that for every $n$ there exists a 1-intersecting family of $n$ distinct subsets of $[n]$. ("Family" is just another word for "set system," i.e., a list of sets.)

Exercise 1.3.26. (a) Construct a 1-intersecting family of 7 3-subsets of $[7]$. (A “3-subset” is a set of 3 elements.) (b) For every prime $p$, construct a 1-intersecting family of $p^2 + p + 1$ $(p + 1)$-subsets of $[p^2 + p + 1]$.

Exercise 1.3.27. * (Generalized Fisher Inequality) Prove: if $k \geq 1$ and $A_1, \ldots, A_m$ is a $k$-intersecting family of distinct subsets of $[n]$ then $m \leq n$. (HINT. First Miracle.)

Exercise 1.3.28. (a) Construct a set of $n$ points in $\mathbb{R}^n$ such that each pair is at the same distance. (b) Construct a set of $n + 1$ points in $\mathbb{R}^n$ such that each pair is at the same distance. (c) Prove: there is no set of $n + 2$ points in $\mathbb{R}^n$ such that each pair is at the same distance.

A “2-distance set” is a set of points with only two distinct distances between them. Example: the regular pentagon in the plane.

Exercise 1.3.29. Prove: a 2-distance set in the plane has at most 5 points.

Exercise 1.3.30. Find a 2-distance set of quadratic size (at least $cn^2$ for some positive constant $c$) in $\mathbb{R}^n$.

Exercise 1.3.31. * Prove: the size of a 2-distance set in $\mathbb{R}^n$ is at most quadratic ($\leq Cn^2$ for some constant $C$). (HINT. First Miracle.)
1.4 Basis

**Definition 1.4.1.** A basis of $V$ is a linearly independent set of generators of $V$, i.e., it is a linearly independent set that spans $V$.

**Definition 1.4.2.** A basis of $S \subseteq V$ is a linearly independent subset of $S$ on which $S$ depends. In other words, a basis $B$ of $S$ is a linearly independent set satisfying $B \subseteq S \subseteq \text{Span}(B)$.

**Exercise 1.4.3.** A list $B$ of vectors is a basis of $V$ iff every vector can be written uniquely as a linear combination of $B$.

**Definition 1.4.4.** Let $B = (b_1, \ldots, b_n)$ be a basis of $V$. Write $x \in V$ as $x = \sum_{i=1}^{n} \beta_i b_i$. The coefficients in this unique linear combination are called the coordinates of $x$ with respect to $B$.

**Definition 1.4.5.** For a basis $B$ of $V$, regarded as a list of vectors, we associate with each $x \in V$ the column vector

$$[x]_B := \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$$

where the $\beta_i$ are the coordinates of $x$ with respect to $B$.

**Exercise 1.4.6.** $B$ is a basis of $S$ if and only if $B$ is a maximal independent subset of $S$.

In view of Ex. 1.3.5 we have the following corollary.

**Corollary 1.4.7.** Every vector space has a basis. In fact, every subset of a vector space has a basis. (Note: in the infinite dimensional case the proof requires Zorn’s Lemma.)

**Exercise 1.4.8.** Show that if the vectors of $B \subseteq S$ are linearly independent, then $B$ can be extended to a basis of $S$.

**Exercise 1.4.9.** Show that every set of generators of $V$ contains a basis. In fact this is true for any subset $S \subseteq V$: if $T \subseteq S$ such that $T$ generates $S$, i.e., $S \subseteq \text{Span}(T)$, then there exists a basis $B$ of $S$ such that $B \subseteq T$.

**Exercise 1.4.10.** Prove: “All bases for $S$ have equal size.” Show that the this statement is equivalent to the “First Miracle”, Theorem 1.3.15.

**Exercise 1.4.11.** Prove: if $B$ is a basis of $V$ then $\dim(V) = |B|$.

**Exercise 1.4.12.** A “Fibonacci-type sequence” is a sequence $(a_0, a_1, a_2, \ldots)$ such that

$$(\forall n)(a_{n+2} = a_{n+1} + a_n).$$

(a) Prove that the Fibonacci-type sequences form a 2-dimensional vector space.

(b) Find a basis in this space consisting of two geometric progressions.

(c) Express the Fibonacci sequence $(0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ as a linear combination of the basis found in item (b).
1.5 Linear maps, isomorphism of vector spaces

Definition 1.5.1. Let $V$ and $W$ be vector spaces. We say that a map $f : V \to W$ is a **homomorphism** or a **linear map** if

(a) $(\forall x, y \in V)(f(x + y) = f(x) + f(y))$
(b) $(\forall x \in V)(\forall \alpha \in \mathbb{R})(f(\alpha x) = \alpha f(x))$

Exercise 1.5.2. Show that if $f$ is a linear map then $f(0) = 0$.

Exercise 1.5.3. Show that $f(\sum_{i=1}^{k} \alpha_i v_i) = \sum_{i=1}^{k} \alpha_i f(v_i)$.

Definition 1.5.4. We say that $f$ is an **isomorphism** if $f$ is a bijective homomorphism.

Definition 1.5.5. Two spaces $V$ and $W$ are **isomorphic** if there exists an isomorphism between them.

Exercise 1.5.6. Show the relation of being isomorphic is an equivalence relation.

Exercise 1.5.7. Show that an isomorphism maps bases to bases. Therefore, isomorphic vector spaces have the same dimension.

Theorem 1.5.8. If $\dim(V) = n$ then $V \cong \mathbb{R}^n$.

Proof: Choose a basis, $B$ of $V$, now map each vector to its coordinate vector, i.e., $v \mapsto [v]_B$.

Exercise 1.5.9. Prove: two vector spaces are isomorphic if and only if they have the same dimension.

Exercise 1.5.10. (Degree of freedom in choosing a linear map) Let $V, W$ be vector spaces. Let $B = (b_1, \ldots, b_k)$ be a basis of $V$. Let $w_1, \ldots, w_k$ be arbitrary vectors in $W$. Prove: there is a unique linear map $f : V \to W$ such that $(\forall i)(f(b_i) = w_i)$.

Exercise 1.5.11. Show that the linear maps $V \to W$ form a vector space. This space is called $\text{Hom}(V,W)$. Prove that its dimension is $\dim(V) \dim(W)$.

Definition 1.5.12. The **image** of $f$ is the set

$$\text{Im}(f) = \{ f(x) : x \in V \}$$

Definition 1.5.13. The **kernel** of $f$ is the set

$$\text{Ker}(f) = \{ x \in V : f(x) = 0 \}$$

Exercise 1.5.14. For a linear map $f : V \to W$ show that $\text{Im}(f) \leq W$ and $\text{Ker}(f) \leq V$.

Theorem 1.5.15. For a linear map $f : V \to W$ we have

$$\dim \text{Ker}(f) + \dim \text{Im}(f) = \dim V.$$

Lemma 1.5.16. If $U \leq V$ and $A$ is a basis of $U$ then $A$ can be extended to a basis of $V$.

Exercise 1.5.17. Prove Theorem 1.5.15. Hint: apply Lemma 1.5.16 setting $U = \text{Ker}(f)$. 

9
1.6 Proof of the First Miracle

Lemma 1.6.1 (Exchange Principle). If \( v_1, \ldots, v_k \) are linearly independent and \( v_1, \ldots, v_k \in \text{Span}(w_1, \ldots, w_\ell) \) then \( \exists j, 1 \leq j \leq \ell \) such that \( w_j, v_2, \ldots, v_k \) are linearly independent. (Note in particular that \( w_j \neq v_2, \ldots, v_k \).)

Exercise 1.6.2. Prove the Exchange Principle.

Exercise 1.6.3. If \( a_1, \ldots, a_k \) are linearly independent and \( (\forall i)(a_i \in \text{Span}\{b_1, \ldots, b_\ell\}) \) then \( (\forall i)(\exists j)(a_1, \ldots, a_{i-1}, b_j, a_{i+1}, \ldots, a_k) \) are linearly independent.

Exercise 1.6.4. Prove the First Miracle, Theorem 1.3.15, using the preceding exercises.

Here is another, more commonly used exchange principle from which the equality of bases and therefore the First Miracle follows.

Theorem 1.6.5 (Steinitz Exchange Principle). Let \( B \) be a basis of \( V \), and let \( L \) be a linearly independent set of vectors in \( V \). Then there exists a subset \( M \subseteq S \) such that \( |M| = |L| \) and \( (S \setminus M) \cup L \) is also a basis of \( V \).

Exercise 1.6.6. Prove the Steinitz Exchange Principle. Do not use consequences of the First Miracle.

Exercise 1.6.7. Use the Steinitz Exchange Principle to prove that all bases have equal size. Infer the First Miracle.

1.7 Matrix rank; the Second Miracle

Definition 1.7.1. An \( m \times n \) matrix is an \( m \times n \) array of numbers \( \{\alpha_{ij}\} \) which we write as

\[
\begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{pmatrix}
\]

Definition 1.7.2. The row rank (or column rank) of a matrix is the rank of the list of its row vectors (column vectors, respectively).

Theorem 1.7.3 (Second Miracle of Linear Algebra). The row rank of a matrix is equal to its column rank.

Definition 1.7.4. The following actions on a set of vectors \( \{v_1, \ldots, v_k\} \) are called elementary operations:

(a) Replace \( v_i \) by \( v_i - \alpha v_j \) where \( i \neq j \).

(b) Replace \( v_i \) by \( \alpha v_i \) where \( \alpha \neq 0 \).
(c) Switch $v_i$ and $v_j$.

**Exercise 1.7.5.** Show that the rank of a list of vectors doesn’t change under elementary operations.

**Exercise 1.7.6.** Let \( \{v_1, \ldots, v_k\} \) have rank \( r \). Show that by a sequence of elementary operations we can get from \( \{v_1, \ldots, v_k\} \) to a set \( \{w_1, \ldots, w_k\} \) such that \( w_1, \ldots, w_r \) are linearly independent and \( w_{r+1} = \cdots = w_k = 0 \).

Consider a matrix. An **elementary row-operation** is an elementary operation applied to the rows of the matrix. Elementary column operations are defined analogously. Exercise 1.7.5 shows that elementary **row-operations** do not change the **row-rank** of \( A \).

**Exercise 1.7.7.** Show that elementary **row-operations** do not change the **column-rank** of a matrix.

**Exercise 1.7.8.** Use Exercises 1.7.5 and 1.7.7 to prove the Second Miracle.