Problem 180. Suppose \( f(x) = x^4 + ax^3 + bx^2 + cx + 15 \) with integer coefficients, i.e., \( a, b, c \in \mathbb{Z} \). Suppose \( k \in \mathbb{Z} \) is a root of \( f(x) \), i.e., \( f(k) = 0 \). What values could \( k \) be? Narrow down the possibilities to a finite number of cases, independent of \( a, b, c \).

Observation 1.3. The complete graph on \( n \) vertices, denoted by \( K_n \), has \( \binom{n}{2} \) edges.

Observation 1.4. Given \( n \) vertices, there are \( 2^{\binom{n}{2}} \) different possible graphs on these \( n \) vertices.

Observation 1.5. An undirected graph on \( n \) vertices has symmetric adjacency matrix and thus diagonalizable with \( n \) real eigenvalues, denoted by \( \lambda_1 \geq \cdots \geq \lambda_n \).

Problem 178. Let \( A, B \in M_n(\mathbb{C}) \). Assume \( AB = BA \). Prove that they have a common eigenvector.

Problem 179. Let \( A, B \in M_n(\mathbb{R}) \), \( A = A^t, B = B^t \) and \( AB = BA \). Prove that they have a common orthonormal eigenbasis.

Exercise 1.1. Suppose \( f(x) = x^4 + ax^3 + bx^2 + cx + 15 \) with integer coefficients, i.e., \( a, b, c \in \mathbb{Z} \). Suppose \( k \in \mathbb{Z} \) is a root of \( f(x) \), i.e., \( f(k) = 0 \). What values could \( k \) be? Narrow down the possibilities to a finite number of cases, independent of \( a, b, c \).

Problem 180. Suppose \( f(x), g(x) \in \mathbb{Z}[x] \) and \( g(x) \) has leading coefficient 1. Prove the division \( f(x) = g(x)q(x) + r(x) \) has integer coefficients quotient and remainder, i.e., \( q(x), r(x) \in \mathbb{Z}[x] \).

Observation 1.6. \( A_G \) denotes the adjacency matrix of the graph \( G \) and \( f_G := f_{A_G} \) denotes the characteristic polynomial of the adjacency matrix \( A_G \).

Exercise 1.7. If \( G, H \) are isomorphic graphs, then \( A_G \) is similar to \( A_H \). In particular, \( f_G = f_H \).

Observation 1.8.

\[
A_{K_n} = \begin{pmatrix}
0 & 1 & \cdots & 1 \\
1 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0
\end{pmatrix} = J_n - I_n,
\]

where

\[
J_n = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}
\]

Observation 1.9. Suppose an \( n \times n \) matrix \( A \) has eigenvalues \( \lambda_1, \ldots, \lambda_n \) (listed with multiplicity), then the matrix \( A - I \) has eigenvalues \( \lambda_1 - 1, \ldots, \lambda_n - 1 \) with the same multiplicity for each eigenvalue of \( A \).

Observation 1.10. \( J_n \) has \( n - 1 \) dimensional null space and thus has eigenvalue 0 with (geometric) multiplicity \( n - 1 \). The remaining eigenvalue is \( n \), using the trace of \( J_n \). Hence, \( f_{J_n}(t) = t^{n-1}(t - n) \) and thus \( f_{K_n}(t) = (t + 1)^{n-1}(t + 1 - n) \).
Problem 182. Suppose $A$ has eigenvalues $\lambda_1, \cdots, \lambda_n$. Prove that $aA + bI$ has eigenvalues $a\lambda_i + b$ with corresponding multiplicities.

Observation 1.11. Suppose $B = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & \vdots \\ \vdots & \vdots & \ddots & b \\ b & \cdots & b & a \end{pmatrix} = bJ_n + (a - b)I_n$, then $B$ has eigenvalue $a - b$ with multiplicity $n - 1$ and another eigenvalue $(n - 1)b + a$ with multiplicity 1. Hence, $f_B(t) = (t - (a - b))^{n-1}(t - (n-1)b - a)$.

Observation 1.12. If $G$ is a regular graph of degree $r$ (every vertex has degree $r$), then $r$ is an eigenvalue with eigenvector $(1, \ldots, 1)^t$.

Problem 183. Assume $A$ is a nonnegative matrix with a positive eigenvector $x$ (all coordinates of $x$ are positive) with eigenvalue $\lambda$, i.e., $x \neq 0$ and $Ax = \lambda x$. Prove $(\forall$ eigenvalue $\mu)(|\mu| \leq \lambda)$.

Exercise 1.13. If a nonnegative symmetric matrix has a positive eigenvector, then all eigenvectors corresponding to other eigenvalues have some negative coordinates.

Exercise 1.14. If $x$ is a nonnegative eigenvector of the connected graph $G$ then $x$ is strictly positive.

Problem 184. Suppose an undirected graph has sorted eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Prove

(1) $(\forall i)(|\lambda_i| \leq \lambda_1)$
(2) If the graph $G$ is connected, then $(\forall i \geq 2)(\lambda_i < \lambda_1)$
(3) If the graph $G$ is connected, then $|\lambda_n| = \lambda_1$ iff $G$ is bipartite.
(4) If $G$ is a bipartite graph, then $(\forall i)(\lambda_i = -\lambda_n - i + 1)$.

Problem 185. Let $g \in \mathbb{C}[x]$ and $A \in M_n(\mathbb{C})$. Assume $A$ has eigenvalues $\lambda_1, \cdots, \lambda_n$ (listed with multiplicity, i.e., $f_A(t) = \Pi_{i=1}^{n}(t - \lambda_i)$). Prove that the eigenvalues of $g(A)$ are $g(\lambda_1), \cdots, g(\lambda_n)$ (again, listed with multiplicity).

Recall that we proved before, if a regular $G$ with degree $r$ has girth at least 5, then $n \geq r^2 + 1$. For such graph, if $a$ and $b$ are two vertices that are not connected, then they share a unique common neighbor. Next we have this amazing theorem.

Theorem 1.15 (Hoffman-Singleton). If a regular graph of degree $r \geq 1$ has girth at least 5 and $n = r^2 + 1$, then we can only have $r = \{1, 2, 3, 7, 57\}$

Observation 1.16. $K_2$ represents $r = 1$. $C_5$ is the example for $r = 2$. Petersen’s graph demonstrates the case $r = 3$. The “Hoffman–Singleton graph” shows $r = 7$ is possible. No example has been found for the case $r = 51$. It remains open.