Supplementary problem set 2 (posted July 6)
REU 2012
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Problem 1. Show that the eigenvectors for distinct eigenvalues are linearly independent.

Problem 2. Show that if A is an $n \times n$ matrix with n distinct eigenvalues, then it is diagonalizable.

Problem 3. Show that the geometric multiplicity of an eigenvalue is at most the algebraic multiplicity of that eigenvalue.

Problem 4. Show that the following are equivalent:

1. A is diagonalizable
2. There is a basis of eigenvectors of A
3. For each eigenvalue λ of A, the geometric multiplicity of λ is equal to its algebraic multiplicity

Problem 5. Find an $n \times n$ matrix with an eigenvalue of algebraic multiplicity n but geometric multiplicity 1.

Problem 6. Show that if A and B are similar matrices, then their characteristic polynomials are equal.

Problem 7. Let A be an invertible integer matrix. Show that A^{-1} is an integer matrix if and only if $\det A = \pm 1$.

Problem 8. Prove the Cayley-Hamilton theorem for diagonal matrices. That is, show that if A is a matrix and $f_A(x)$ is the characteristic polynomial of A, then $f_A(A) = 0$.

Problem 9. Prove the Cayley-Hamilton theorem for diagonalizable matrices. *Hint: first prove that if A and B are similar and f is a polynomial, then $f(A)$ and $f(B)$ are similar.*

Problem 10. Prove that diagonalizable matrices are dense in $A_n(\mathbb{C})$.

Problem 11. Infer from the previous problems that the Cayley-Hamilton theorem is also true over $\mathbb{C}$. Note in particular that it is true over $\mathbb{Z}$.

Problem 12. Prove from the last problem (over $\mathbb{Z}$) that the Cayley-Hamilton theorem is true over every field (and, in fact, over every commutative ring with identity).

Problem 13. The *minimal polynomial* $m_A(x)$ for a matrix A is the unique polynomial $m(x)$ of minimum degree, whose leading coefficient is 1, such that $m(A) = 0$. Show that the minimal polynomial of a matrix divides its characteristic polynomial.

Problem 14. Prove that λ is an eigenvalue of A if and only if $m_A(\lambda) = 0$.

Problem 15. Prove that a matrix $A \in M_n(\mathbb{C})$ is diagonalizable if and only if the minimal polynomial of A has no repeated roots.

Problem 16. Find an $n \times n$ matrix whose minimal polynomial is x^n.

Problem 17. Let $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x]$. Assume that $f(r/s) = 0$, where r/s is in lowest terms. Show that $r|a_0$ and $s|a_n$.

Problem 18. (Triangle Inequality.)

Prove that for any vectors v, w in a real Euclidean space, we have

$$\|v + w\| \leq \|v\| + \|w\|.$$

Definiton 1. Let φ be a linear transpormation of the Euclidean space V. Recall that φ is *symmetric* if

$$\langle x, \varphi y\rangle = \langle \varphi x, y\rangle$$

for all x and y.

Problem 19. Fix an orthonormal basis. Show that a linear map φ is symmetric if and only if its matrix with respect to the orthonormal basis is symmetric.

Definiton 2. Let V be a Euclidean space and $\varphi : V \to V$ a symmetric linear transformation. The *Rayleigh Quotient* of φ is defined by

$$R_\varphi(x) = \frac{\langle x, \varphi x\rangle}{\|x\|^2}.$$

where $x \in V$, $x \neq 0$.

Problem 20. Recall Rayleigh's theorem from class:

$$\lambda_1 = \max_{x \neq 0} R_\varphi(x)$$

where λ_1 is the greatest eigenvalue for φ. Prove: $\lambda_n = \min_{x \neq 0} R_\varphi(x)$.

Problem 21. (Courant-Fischer theorem) Let φ be a symmetric transformation of a real Euclidean space. Prove: if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues for φ then

$$\lambda_i = \max_{\substack{U \leq V \\ \dim U = i}} \min_{\substack{x \in U \\ x \neq 0}} R_\varphi(x).$$

Problem 22. (Interlacing theorem) Let A be a symmetric $n \times n$ matrix over $\mathbb{R}$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues for A. Form the symmetric $(n-1) \times (n-1)$ matrix B by removing the jth row and the jth column from A, and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ be its eigenvalues.

Show that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

(Use the Courant-Fischer theorem.)