11 Eleventh Class: Fri. 7/18/14

11.1 First session: Review of graph theory

Grötzsch’s Graph: A graph $G \not\supset K_3$ (triangle-free) such that $\chi(G) \geq 4$ (not 3-colorable). *Hint:* $11$ vertices, 5-fold symmetry.

Mantel-Turán Theorem: If $G \not\supset K_3$ then $m \leq \frac{n^2}{4}$ where $m$ is the number of edges and $n$ the number of vertices.

Bipartite graph: A graph whose vertex set can be split into two nonempty subsets such that no vertex is adjacent to any other vertex in its subset. For a bipartite graph $G$, we know that $\chi(G) \leq 2$, and we say that $G$ is “two-colorable.” This is equivalent to saying that a bipartite graph includes no odd cycles. It is also obvious that bipartiteness of $G$ implies $m \leq \frac{n^2}{4}$, since if one subset is of cardinality $r$ and the other of cardinality $s = n - r$, then $m \leq rs = r(n - r) \leq \frac{n^2}{4}$.

Complete bipartite graph: $K_{r,s}$, the complete bipartite graph, has $r + s$ vertices and $r \cdot s$ edges.

Kőnig-Turán-Sós Theorem: If $G \not\supset C_4$ then $m < cn^{\frac{3}{2}}$. This bound is tight: There exists a $c > 0$ such that, for all $n$, there exists $G \not\supset C_4$ with $m \geq cn^{\frac{3}{2}}$.

Digression: If you have a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx - 15$ where $a, b, c \in \mathbb{Z}$, then the rational roots of this polynomial are in what small set? First, can we limit the possible values of integer roots? Well, if $x$ is a root, then $x^4 + ax^3 + bx^2 + cx - 15 = 0 \Rightarrow x^4 + ax^3 + bx^2 + cx = 15 \Rightarrow x(x^3 + ax^2 + bx + c) = 15 \Rightarrow x \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$. Now, prove that this set is a limiting set on rational roots, as well. That is, prove that a rational root is necessarily an integer. To do this, write $x = \frac{r}{s}$, assume $\gcd(r, s) = 1$, and prove $s = 1$.

Girth: The girth of a graph is the length of the shortest cycle. A tree is a connected graph with no cycles. (The girth of a tree is defined to be infinite.) The girth of a graph $G$ is $\geq k$ if $G$ does not contain any cycle of length $< k$. We say that a graph is connected if there exists a walk between any pair of vertices. For any graph $G$ and any vertices $x, y \in V(G)$, we can define

$$R(x, y) = \begin{cases} 1 & \exists x \cdots y \text{ walk,} \\ 0 & \text{otherwise.} \end{cases}$$

This is called a predicate; a predicate on pairs is called a relation; a predicate on a set is called a subset. $R$ is an equivalence relation, which means the following hold:

- $R(x, x)$ reflexive
• \( R(x, y) \iff R(y, x) \) symmetric

• \( R(x, y) \) and \( R(y, z) \iff R(x, z) \) transitive.

The division of a set \( \Omega \) into disjoint subset blocks \( A_i \neq \emptyset \) satisfying \( \Omega = A_1 \cup \cdots \cup A_k \) is called a partition. Out of any partition, we can make an equivalence relation by declaring \( R_x(x, y) \) \( \overset{\text{def}}{=} (\exists i)(x, y \in A_i) \).

**Exercise 11.1.** If \( R \) is an equivalence relation on \( \Omega \), then there exists a unique partition \( \pi \) of \( \Omega \) such that \( R = R_\pi \).

In constructing \( \mathbb{Q} \) from \( \mathbb{Z} \), we need to create an equivalence relation on “fractions”: We say that two fractions \( \frac{r}{s} \) and \( \frac{x}{y} \) are equivalent if \( ry = xs \).

**Exercise 11.2.** This is an equivalence relation on “fractions.”

The resulting equivalence classes of fractions are called “rational numbers.” Using the equivalence relation on graph vertices above, we can observe that the connected components of a graph are the blocks of the equivalence relation “\( x, y \) are connected by a walk.” In this sense, a connected graph is a graph with only one connected component. A tree is then a connected graph with no cycles. A forest is a graph with no cycles whose connected components are trees.

**Exercise 11.3.** For a tree, \( m = n - 1 \).

**Exercise 11.4.** If \( G \) has \( k \) connected components, then \( m \geq n - k \).

If a graph \( G \) has girth \( \geq 5 \) and is \( k \)-regular \( ((\forall x)(\deg(x) = k)) \), then \( n \geq k^2 + 1 \).

**Proof.** Start at one vertex \( v_0 \); it has degree \( k \), so we need find only \( k^2 - k = k(k - 1) \) more vertices. Each of the \( k \) neighbors of \( v_0 \) has \( k - 1 \) other neighbors; these \( v \sim v_0 \) cannot share neighbors, since then the girth of the graph would be \( \leq 4 \). Hence these are distinct, and we have found \( k^2 + 1 \) vertices.

We discussed that equality sometimes holds in \( n \geq k^2 + 1 \); in particular, it holds for \( k = 1 \) (\( K_2 \)), for \( k = 2 \) (\( C_5 \)), for \( k = 3 \) (Petersen’s graph), for \( k = 7 \) (the Hoffman-Singleton graph).

**Hoffman-Singleton Theorem:** If girth(\( G \)) \( \geq 5 \) and \( G \) is \( k \)-regular and \( n = k^2 + 1 \), then \( k \in \{1, 2, 3, 7, 57\} \).

Let \( G \) be a graph satisfying these conditions. If \( x \neq y \in V(G) \) and \( x \sim y \) (\( x \) and \( y \) are not adjacent), then it follows that \( x \) and \( y \) have exactly one common neighbor.

**Proof.** This follows from the proof above; let \( v_0 = x \). \( \square \)

### 11.2 Second session: Proof of Hoffman-Singleton Theorem, Hermitian space

Let \( A = A_G = (a_{ij})_{n \times n} \) be the adjacency matrix, where

\[
a_{ij} = \begin{cases} 
1 & i \sim j, \\
0 & i \not\sim j.
\end{cases}
\]

As discussed on Wednesday, \( A^2 = (b_{ij}) \) where \( b_{ij} \) equals the number of common neighbors of \( i \) and \( j \). In a graph \( G \) satisfying the Hoffman-Singleton Theorem, the following holds:

\[
b_{ij} = \begin{cases} 
k & i = j, \\
0 & i \sim j \text{ because of the girth condition,} \\
1 & i \not\sim j \text{ by proof above.}
\end{cases}
\]

That is, in this case,

\[
A^2 = kI + A_G^2 = kI + J - I - A = (k - 1)I + J - A.
\]
Exercise 11.5. Let $J$ be the all-ones matrix. Then $A_G + A_G = J - I$, since our definition of graph (and hence definition of graph complement) excludes loops.

Hence our matrix satisfies

$$A^2 + A = (k - 1)I = J.$$  

(1)

In the adjacency matrix, the degree of the $i$th vertex is the sum of the entries in the $i$th row, generally. In this case, every row sums to $k$. That is,

$$A_G \cdot 1 = A_G \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = k \cdot 1.$$  

This shows us that:

Exercise 11.6. The all-ones vector $1 \in \mathbb{R}^n$ is an eigenvector of the adjacency matrix of $G$, with corresponding eigenvalue $k$ iff $G$ is $k$-regular.

This means that $A$ is a symmetric real matrix. By the spectral theorem, we then have an orthonormal eigenbasis. Therefore every orthonormal list of eigenvectors can be extended to an orthonormal eigenbasis. Since $||1|| = \sqrt{n}$, we take $b_1 = \frac{1}{\sqrt{n}} \cdot 1, 1, \ldots, 1$. For $i \geq 2, b_1 \perp 1$. In a matrix form, this means that $1^t \cdot b_i = 0$ under the standard dot product. Of course, $Ab_1 = \lambda_1 b_1$ for some $\lambda_1 \in \mathbb{R}$. We know that $\lambda_1 = k$.

Furthermore, we know that $A^2 b_i = AAb_i = A\lambda_1 b_i = \lambda_i A b_i = \lambda_i^2 b_i$. We will also need to multiply $Ib_i = b_i$.

Finally, we know that $Jb_i = 0$ for $i \geq 2$, since $b_i \perp 1$.

Now we can work with Equation 1. For $i \geq 2$:

$$(\lambda_i^2 + \lambda_i - (k - 1)) b_i = 0$$

$$(\lambda_i^2 + \lambda_i - (k - 1)) b_i = 0$$

$$(\lambda_i^2 + \lambda_i - (k - 1)) b_i = 0$$

$$\lambda_i = \frac{-1 \pm \sqrt{1 + 4(k - 1)}}{2} = \frac{1 \pm \sqrt{4k - 3}}{2}.$$  

If we define $s = \sqrt{4k - 3}$, then $s^2 = 4k - 3 \implies k = \frac{s^2 + 3}{4}$.

Exercise 11.7. For $m \in \mathbb{N}$, either $\sqrt{m} \in \mathbb{Z}$ or $\sqrt{m} \in \mathbb{R} \setminus \mathbb{Q}$.

So the eigenvalues and their multiplicities:

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Multiplicities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_1 = \frac{-1 + s}{2}$</td>
<td>$m_1$</td>
</tr>
<tr>
<td>$\lambda_2 = \frac{-1 - s}{2}$</td>
<td>$m_2$</td>
</tr>
</tbody>
</table>

We know that

$$1 + m_1 + m_2 = n = k^2 + 1.$$  

But we need more information, so we can use the trace. For a matrix $A$, we know that $\text{Tr}(A) = \sum a_{ii} = \sum \lambda_i$. The trace of the adjacency matrix of a graph is zero. Hence we also know that

$$k + m_1 \lambda_1 + m_2 \lambda_2 = 0.$$  

Now we have two equations on two unknown multiplicities. Simplifying, we have

$$m_1 + m_2 = k^2.$$  

3
and

\[ k - \frac{1}{2}(m_1 + m_2) + \frac{1}{2}s(m_1 - m_2) = 0 \]
\[ 2k - k^2 + s(m_1 - m_2) = 0 \]
\[ s(m_1 - m_2) = k^2 - 2k. \]

By the exercise, we know that \( s \) is either an integer or irrational.

- **Case 1** If \( s \) is irrational, then the equation can only hold if \( m_1 - m_2 = 0 \implies m_1 = m_2 \) and \( k^2 - 2k = 0 \implies k = 2 \), which means we are discussing \( C_5 \).

- **Case 2** If \( s \) is an integer, then we want to show that \( k \in \{1, 2, 3, 57\} \). Since \( k = \frac{s^2 + 3}{4} \) is an integer, we need \( s \) to be odd. This rules out \( k = 4, 5, 6, 8, \ldots, 12, 14, \ldots, 20 \). The overwhelming majority of numbers is thus ruled out. Plugging in, we obtain

\[ s(m_1 - m_2) = \frac{(s^2 + 3)^2}{2} - 2\left(\frac{s^2 + 3}{4}\right) \]
\[ 16s(m_1 - m_2) = (s^2 + 3)^2 - 8(s^2 + 3) \]
\[ = s^4 + 6s^2 + 9 - 8s^2 - 24 \]
\[ = s^4 - 2s^2 - 15 \]
\[ \implies s^4 - 2s^2 - 16(m_1 - m_2)s - 15 = 0. \]

By the digression above, we have \( s = \pm 1, \pm 3, \pm 5, \pm 15 \implies \frac{s^2 + 3}{4} = 1, 3, 7, 57 \).

**Hermitian dot product:** For two vectors \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \) and \( y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \) in \( \mathbb{C}^n \), we define the Hermitian dot product as

\[ x^* \cdot y = \sum x_i \overline{y_i} \]

where \( \overline{y_i} \) is the complex conjugate of \( y_i \). Note that \( x^* \cdot x = \sum \overline{x_i}x_i = \sum |x_i|^2 \in \mathbb{R}, > 0 \) unless \( x = 0 \).

Note the following use of scalars:

\[ x^* \cdot \lambda y = \lambda (x^* \cdot y), \]
\[ (\lambda x)^* \cdot y = \overline{\lambda}(x^* \cdot y). \]

For this reason, the Hermitian dot product is not called bilinear. Instead, it is called *sesquilinear*.

**Hermitian space:** A *Hermitian space* is a vector space \( V \) over \( \mathbb{C} \) with a sesquilinear, positive-definite inner product:

1. \((x, y) \in V \times V \implies \exists \ < x, y > \in \mathbb{C} \)
2. (a) \(< x + y, z > = < x, z > + < y, z > \)
   (b) \(< x, y + z > = < x, y > + < x, z > \)
3. (a) \(< x, \lambda y > = \lambda < x, y > \)
   (b) \(< \lambda x, y > = \overline{\lambda} < x, y > \) (1, 2, 3 define a sesquilinear product \(< \cdot, \cdot > \))
4. \(< y, x > = \overline{< x, y >} \implies < x, x > = \overline{< x, x >} \in \mathbb{R} \) (Hermitian symmetry)
(5) \((\forall x \neq 0)(< x, x > > 0)\) (positive definiteness)

**Exercise 11.8.** Perform Gram-Schmidt in Hermitian spaces. Corollary: If \(\text{dim} \mathbb{C}\) is finite or countably infinite, then there exists an orthonormal basis. In fact, any set of orthonormal vectors can be extended to an orthonormal basis.

What would correspond to symmetric matrices in this context? A Hermitian matrix should be a matrix satisfying \(A^* = A\) where \(A^* = \overline{A} = (\pi_{ij})\).

**Exercise 11.9.** If \(A\) is Hermitian, then every eigenvalue of \(A\) is real.

**Abridged Complex Spectral Theorem:** If \(A = A^*\), then there exists an orthonormal eigenbasis and all eigenvalues are real. We know that an orthogonal matrix in \(M_n(\mathbb{R})\) is a matrix whose columns form an orthonormal basis; that is, a matrix \(A\) satisfying \(A^tA = I\). The analogue in Hermitian spaces is the unitary matrix: A matrix \(A \in M_n(\mathbb{C})\) is called unitary if \(A^*A = I\).

We referred to the fact that orthonormality of the columns of a matrix implies orthonormality of the rows as the Third Miracle of Linear Algebra. Let’s prove that quickly:

**Proof.** The assumption in matrix equation form is that \(A^tA = I\). The desired conclusion in matrix equation form is that \(AA^t = I\). The question is whether \(A\) commutes with \(A^t\). The given condition means that \(A^t = A^{-1}\). Generally, \(A^{-1}\) exists iff \(A\) has full column rank, and \(A^{-1}\) exists iff \(A\) has full row rank. By the Second Miracle (the row-rank is equal to the column rank), if \(A\) is a square matrix in \(M_n(\mathbb{F})\), then these two are equivalent, \(\exists A^{-1}_{\text{left}} \iff \exists A^{-1}_{\text{right}}\), and then it follows that the left inverse is equal to the right inverse, since \(\exists X : XA = I\) and \(\exists Y : AY = I \implies X = X(AY) = (XA)Y = Y\). Hence the Third Miracle follows from the second miracle.

A matrix is unitary if \(A^* = A^{-1} \implies AA^* = I\).

**Exercise 11.10.** If \(A\) is unitary and \(\lambda\) is an eigenvalue of \(A\), then \(|\lambda| = 1\).

**Definition:** A normal matrix is a matrix \(A \in M_n(\mathbb{C})\) such that \(AA^* = A^*A\). Special cases of this are unitary matrices and Hermitian matrices.

**Full Complex Spectral Theorem:** If \(A \in M_n(\mathbb{C})\), then \(A\) is normal iff there exists an orthonormal eigenbasis.

**Exercise 11.11.** If \(A \in M_n(\mathbb{C})\) and \(A\) is triangular and normal, then \(A\) is diagonal. Note that

\[
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & \lambda_{n-1} & 0 \\
0 & \cdots & 0 & \lambda_n
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & \lambda_{n-1} & 0 \\
0 & \cdots & 0 & \lambda_n
\end{pmatrix}
= \begin{pmatrix}
|\lambda_1|^2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & |\lambda_{n-1}|^2 & 0 \\
0 & \cdots & 0 & |\lambda_n|^2
\end{pmatrix}.
\]