In these notes, \( \mathbb{F} \) denotes a field. The most important examples we need are \( \mathbb{F} = \mathbb{C} \) and \( \mathbb{F} = \mathbb{R} \), so you can just think of these two. Other important cases include \( \mathbb{F} = \mathbb{Q} \) and \( \mathbb{F} = \mathbb{F}_p \). Some of the results only work over \( \mathbb{C} \), however.

**Definition.** A (univariate) polynomial \( f \in \mathbb{F}[x] \) (over the field \( \mathbb{F} \)) is given by the formal expression \( f(x) = \sum_{k=0}^{n} a_k x^k \). The \( a_k \in \mathbb{F} \) are called the coefficients of \( f \). Two polynomials are equal if they have the same coefficients (except that zero coefficients need not be written). The degree of \( f \) is \( \deg(f) = \max\{j : a_j \neq 0\} \). In particular, the degree of the zero polynomial is \( \deg(0) := \max\emptyset = -\infty \).

**Exercise 13.1.** Show that
(a) \( \deg(f + g) \leq \max\{\deg f, \deg g\} \).
(b) \( \deg(f \cdot g) = \deg(f) + \deg(g) \).

**Remark.** The value \( \deg(0) = -\infty \) is necessary to preserve properties (a) and (b).

**Definition.** A bivariate polynomial is given by the formal expression \( f(x,y) = \sum a_{ij} x^i y^j \), where \( a_{ij} \in \mathbb{F} \). The degree of \( f \) is given by \( \deg(f) = \max\{i + j : a_{ij} \neq 0\} \). Again, \( \deg(0) := -\infty \). The space of these polynomials is denoted \( \mathbb{F}[x,y] \). The space \( \mathbb{F}[x_1, \ldots, x_k] \) of polynomials in \( k \) variables (multivariate polynomials) and the degree of such polynomials is defined analogously.

**Exercise 13.2.** We write \( \mathbb{F}^{\leq n}[x] \) to denote the space of polynomials of degree \( \leq n \).
(a) Show that \( \dim(\mathbb{F}^{\leq n}[x]) = n + 1 \).
(b) Find \( \dim(\mathbb{F}^{\leq n}[x,y]) \).
(c) Find \( \dim(\mathbb{F}^{\leq n}[x_1, \ldots, x_k]) \). The answer is a binomial coefficient involving \( n \) and \( k \).

**Exercise 13.3.** (Division Theorem for Integers)
For any \( a, b \in \mathbb{Z} \), if \( b \neq 0 \) then \( (\exists q, r \in \mathbb{Z})(a = bq + r \text{ and } 0 \leq r < |b|) \).

**Exercise 13.4.** (Division Theorem in \( \mathbb{F}[x] \))
For any \( f, g \in \mathbb{F}[x] \), if \( g \neq 0 \) then \( (\exists q, r \in \mathbb{F}[x])(f = gq + r \text{ and } \deg(r) < \deg(g)) \).
Exercise 13.5. Show that $f(x) = q(x)(x - \alpha) + f(\alpha)$.

**Definition.** Suppose that $f, g \in \mathbb{F}[x]$, we say that $g$ divides $f$, denoted by $g \mid f$, if $(\exists q \in \mathbb{F}[x])(f = qg)$.

**Remark.** (1) $(\forall g \in \mathbb{F}[x])(g \mid 0)$. (2) $0 \mid f$ iff $f = 0$.

**Definition.** The value $\alpha \in \mathbb{C}$ is a double root of $f \in \mathbb{C}[x]$ if $(x - \alpha)^2 \mid f$. The multiplicity of root $\alpha$ in $f$ is given by $\max\{k : (x - \alpha)^k \mid f\}$.

Exercise 13.6.

(a) $\alpha$ is a multiple root (multiplicity $\geq 2$) of $f \in \mathbb{C}[x]$ iff $f(\alpha) = f'(\alpha) = 0$.

(b) Characterize multiplicity in terms of the derivative.

(c) Prove that $x^{100} - x + 1$ has no multiple roots in $\mathbb{C}$ by hand calculation.

(The proof should be just a few lines.)

**Theorem** (Fundamental Theorem of Algebra).

If $f \in \mathbb{C}[x]$ and $\deg(f) \neq 0$, then $(\exists \alpha \in \mathbb{C})(f(\alpha) = 0)$, i.e., $f$ has a complex root.

**Theorem** (Fundamental Theorem of Algebra V.2).

For all $f = \sum_{i=1}^{n} a_i x^i \in \mathbb{C}[x]$ such that $\deg f = n$, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that $f(x) = a_n \prod_{i=1}^{n} (x - \alpha_i)$.

Example: $x^n - 1 = \prod (x - \omega^i)$, where $\omega = e^{2\pi i/n}$ ($i = \sqrt{-1}$). The roots $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the $n$-th roots of unity.

Exercise 13.7. Prove that the sum of the $n$-th roots of unity is 0.

**Definition.** The order of $z$ is defined as $\text{ord}(z) := \min\{k \in \mathbb{N} : z^k = 1\}$.

Exercise 13.8.

(a) Suppose that $\text{ord}(z) = k$. Show that $(\forall \ell)(z^\ell = 1 \iff k \mid \ell)$.

Hint: $\Leftarrow$ is trivial. $\Rightarrow$ uses the Division Theorem.

(b) Suppose that $\text{ord}(z) = k$. What is $\text{ord}(z^2)$?

Exercise 13.9.

(a) Consider the equilateral triangle inscribed in a unit circle (circle of radius 1). Show that its sides have length $\sqrt{3}$.

(b) $\heartsuit$ Consider a regular $n$-gon inscribed in a unit circle; let its vertices be $A_0, \ldots, A_{n-1}$.

Show that $\prod_{j=1}^{n-1} A_0 A_j = n$.

Here $A_0 A_j$ denotes the distance between $A_0$ and $A_j$.  

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Exercise 13.10. (Rank-Nullity Theorem)
Let \( f : V \to W \) be a homomorphism. Then, \( \dim(\ker(f)) + \dim(\text{im}(f)) = \dim(V) \).

**Theorem.** Consider the homogeneous system of linear equations given by

\[
\begin{align*}
  a_{11}x_1 + \ldots + a_{1n}x_n &= 0 \\
  a_{21}x_1 + \ldots + a_{2n}x_n &= 0 \\
  &\vdots \\
  a_{k1}x_1 + \ldots + a_{kn}x_n &= 0
\end{align*}
\]

\((k \text{ equations in } n \text{ unknowns}). Using matrix notation, this can be written concisely as

\[Ax = 0,\]

where \( A = (a_{ij})_{k \times n} \) and \( x = (x_1, \ldots, x_n)^T \). (The \( T \) indicates “transpose,” so this is a column vector). We define the solution space to be \( U = \{x \in \mathbb{R}^n \mid Ax = 0\} \leq \mathbb{R}^n \). Then

\[\dim U = n - \text{rank}(A).\]

**Proof.** Define the homomorphism \( f : \mathbb{R}^n \to \mathbb{R}^k \) by \( f : x \mapsto Ax \). Then, \( \ker f = U \), so the kernel of \( f \) is the solution space. Also, \( \text{im } f = \text{colspace}(A) \), so \( \dim(\text{im } f) = \text{rank } A \). Use the Rank-Nullity Theorem.

**Exercise 13.11.** Let \( A \in \mathbb{C}^{k \times n} \) and \( B \in \mathbb{C}^{n \times k} \). Show that \( \text{Tr}(AB) = \text{Tr}(BA) \).

**Exercise 13.12.** Let \( A \in M_n(\mathbb{C}) \). The characteristic polynomial of \( A \) is defined by \( f_A(t) := \det(tI - A) \).

(a) Show that \( f_A(t) = t^n - (\text{Tr } A)t^{n-1} + \ldots \pm (\det A) \).

Decide the sign of the last term.

(b) Find the coefficient of \( t^{n-2} \) in \( f_A(t) \).

(c) Generalize.

**Exercise 13.13.** Find the characteristic polynomial of

\[
\begin{pmatrix}
  0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 \\
  5 & 0 & -1 & 7 & 2
\end{pmatrix}.
\]

Generalize this result.

**Definition.** The **algebraic multiplicity** of an eigenvalue \( \lambda \) of \( A \) is the multiplicity of \( (t - \lambda) \) in \( f_A(t) \). The **geometric multiplicity** of \( \lambda \) of \( A \) is the number of linearly independent eigenvectors to \( \lambda \).
Exercise 13.14. For any $\lambda \in \mathbb{C}$, let $U_\lambda = \{x : Ax = \lambda x\} \subseteq \mathbb{C}^n$.

(a) Show that the geometric multiplicity of $A$ is equal to $\dim(U_\lambda)$.

(b) Show that $\dim(U_\lambda) = n - \text{rank}(\lambda I - A)$

Exercise 13.15. Show that $\text{geom-mult}(\lambda) \leq \text{alg-mult}(\lambda)$.

Exercise 13.16. Show that $\det(AB) = \det(A) \cdot \det(B)$.

Exercise 13.17. Show that if $v_1, \ldots, v_k$ are eigenvectors of $A$ to distinct eigenvalues, then they are linearly independent.

Definition. Let $A, B \in M_n(\mathbb{C})$. We say that $A$ and $B$ are similar, denoted $A \sim B$, if $(\exists C \in M_n(\mathbb{C}))(B = C^{-1}AC)$.

Exercise 13.18. Suppose that $A \sim B$. Show that the following hold:

(a) $\text{Tr}(A) = \text{Tr}(B)$.

(b) $\det(A) = \det(B)$.

(c) $f_A(t) = f_B(t)$.

Definition. A matrix $A \in M_n(\mathbb{C})$ is diagonalizable if $A$ is similar to a diagonal matrix.

Definition. An eigenbasis of $A \in M_n(\mathbb{C})$ is a basis of $\mathbb{C}^n$ consisting of eigenvectors of $A$.

Exercise 13.19. Suppose that $A \in M_n(\mathbb{C})$. Show that the following are equivalent:

(a) $A$ is diagonalizable

(b) $A$ has an eigenbasis

(c) For every (complex) eigenvalue $\lambda$ of $A$, $\text{geom-mult}(\lambda) = \text{alg-mult}(\lambda)$.

Exercise 13.20. (a) Show that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable. (Done in class)

(b) Show that $\begin{pmatrix} 1 & 1 \\ 0 & 75 \end{pmatrix}$ is diagonalizable.

(c) Show that $\begin{pmatrix} 7 & s \\ 0 & 7 \end{pmatrix}$ is diagonalizable iff $s = 0$.

Exercise 13.21. Consider the rotation matrix $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. Find a (complex) eigenbasis for $R_\theta$ and find the eigenvalues of $R_\theta$. Note: the eigenbasis will not depend on $\theta$.

Exercise 13.22. Show that if $f_A(t)$ has no multiple roots, then $A$ has an eigenbasis.

Exercise 13.23. If $A \in M_n(\mathbb{C})$, then $A$ is similar to a triangular matrix. Hint: Induction.

Exercise 13.24. If $A \in M_3(\mathbb{R})$, then $A$ has an eigenvector (in $\mathbb{R}^3$).