15.1 Translates, Affine Subspaces, and Systems of Linear Equations

A system of linear equations is given by

\begin{align*}
    a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + \ldots + a_{2n}x_n &= b_2 \\
    \vdots & \quad \vdots \\
    a_{k1}x_1 + \ldots + a_{kn}x_n &= b_k
\end{align*}

This may be written more concisely as

\[ A\vec{x} = \vec{b} \quad (1) \]

where \( A = (a_{ij}) \in \mathbb{F}^{k \times n} \), \( \vec{x} = (x_1,\ldots,x_n)^T \in \mathbb{F}^n \), and \( \vec{b} = (b_1,\ldots,b_k)^T \in \mathbb{F}^k \). Equation (1) is solvable if there exists a satisfying \( \vec{x} \).

**Theorem.** \( A\vec{x} = \vec{b} \) is solvable iff \( \vec{b} \in \text{col-space}(A) \).

**Proof.** If \( A = [\vec{a}_1 \cdots \vec{a}_n] \), Equation (1) becomes \( A\vec{x} = x_1\vec{a}_1 + \ldots + x_n\vec{a}_n = \vec{b} \).

**Exercise 15.1.** \( \exists \vec{x} \) \( (A\vec{x} = \vec{b}) \iff \text{rank}(A) = \text{rank} \left( A|\vec{b} \right) \).

**Definition.** Let \( S \subseteq V \) and \( v \in V \). The translate of \( S \) by \( v \) is \( S + v = \{ s + v : s \in S \} \). An affine subspace is a translate of a subspace. An affine combination of \( v_1,\ldots,v_k \) is \( \sum_i \alpha_i v_i \) such that \( \sum_i \alpha_i = 1 \).

**Definition.** Let \( \text{Aff}(\vec{a},\vec{b}) := \{ \text{affine combinations of } \vec{a} \text{ and } \vec{b} \} \). More generally, for \( S \subseteq V \), let \( \text{Aff}(S) := \{ \text{ all affine combinations of elements of } S \} \). If \( S \) is an infinite set, an affine combination of \( S \) contains only expressions with a finitely many non-zero coefficients. Notice that \( \text{Aff}(\emptyset) = \emptyset \).

**Exercise 15.2.** Associate to a vector \( \vec{a} \) with ‘tail’ at 0 the point \( A \) at the ‘head’ of \( \vec{a} \). We keep this convention throughout this exercise.
(a) Show that Aff($\vec{a}, \vec{b}$) is exactly the line through points $A$ and $B$.

(b) Show that Aff($\vec{a}, \vec{b}, \vec{c}$) is exactly the plane through $A$, $B$, and $C$.

**Exercise 15.3.** Show: for $\emptyset \neq S \subseteq V$ the following are equivalent:

1. $S$ is a translate of a subspace.
2. Aff($S$) = $S$, i.e., $S$ is closed under affine combinations.

**Exercise 15.4.** Show: the intersection of any number of affine subspaces is either an affine subspace or empty.

**Exercise 15.5.** Show: the set $\{x : Ax = b\}$ is either an affine subspace or empty, i.e., any affine combination of solutions is a solution.

**Exercise 15.6.** Let $U = \{x : Ax = 0\}$ and $S = \{x : Ax = b\}$. Show that if $S \neq 0$ then $S$ is a translate of $U$ by any $s \in S$.

**Proof.** In other words, if $As = b$, then $U + s = S$. In other words, if $As = b$, then $(\forall u)(Au = 0 \iff A(u + s) = b)$. □

**Definition.** The *dimension* of an affine subspace $S$ is, for any $s \in S$, given by $\dim(S) = \dim(S - s)$.

**Corollary.** If $Ax = b$ is solvable then the set of solutions is an affine subspace of dimension $n - \text{rank}(A)$.

### 15.2 Transpose

**Definition.** Let $A = (a_{ij})_{k \times n}$. The *transpose* of $A$ is $A^T = (a_{ji})_{n \times k}$.

For example, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$.

**Exercise 15.7.**

(a) Show: $(AB)^T = B^T \cdot A^T$.

(b) Show: $\text{rank}(A^T) = \text{rank}(A)$.

(c) Show: $\det(A^T) = \det(A)$. Hint: use part (d) below.

(d) Show that $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$. 
15.3  \( \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \)

**Proof.** First we show that \( \text{rank}(AB) \leq \text{rank}(A) \). Note that \( \text{rank}(AB) \leq \text{rank}(A) \iff \text{dim}([\text{col-space}(AB)]) \leq \text{dim}([\text{col-space}(A)]) \). But, \( AB = [Ab_1, \ldots, Ab_n] \) and \( Ab_j \in \text{col-space}(A) \). So, each of the columns of \( AB \) is contained in \( \text{col-space}(A) \), so \( \text{col-space}(AB) \leq \text{col-space}(A) \).

Now we show that \( \text{rank}(AB) \leq \text{rank}(B) \).

First Proof: Show that \( \text{row-space}(AB) \leq \text{row-space}(B) \) by the same argument as above. Second Proof: \( \text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^T A^T) \leq \text{rank}(B^T) = \text{rank}(B) \). □

15.4 Right and Left Inverses

**Definition.** Let \( A \in \mathbb{F}^{k \times n} \). A **right inverse** of \( A \) is a matrix \( B \in \mathbb{F}^{n \times k} \) such that \( AB = I_k \).

A **left inverse** of \( A \) is a matrix \( C \in \mathbb{F}^{k \times n} \) such that \( BA = I_k \).

**Theorem.** \( A \) has a right inverse iff \( A \) has full row rank. \( A \) has a left inverse iff \( A \) has full column rank.

**Proof.** If \( B \) is a right inverse of \( A \), then \( AB = [Ab_1 \cdots Ab_k] = [e_1 \cdots e_k] \). Such a \( B \) exists iff \( (\forall i = 1 \ldots k)(\exists b_i)(Ab_i = e_k) \iff (\forall i = 1 \ldots k)(\exists b_i)(e_i \in \text{col-space}(A)) \iff \text{col-space}(A) = \mathbb{F}^k \iff \text{rank}(A) = k \).

The left inverse argument is similar using row rank. □

**Exercise 15.8.** Prove the second half of the above theorem using transposes.

**Exercise 15.9.** If \( A \in \mathbb{F}^{k \times n} \) and \( k < n \), then \( A \) cannot have a left inverse.

**Exercise 15.10.** If \( A \in \mathbb{R}^{k \times n} \) and \( A \) has full row rank, then \( A \) has infinitely many right inverses.

**Exercise 15.11.** Find a basis of \( \mathbb{F}^{k \times n} \). Note that \( \text{dim}(\mathbb{F}^{k \times n}) = k \cdot n \).

**Exercise 15.12.** The set of inverses of \( A \) (if nonempty) is an affine subspace of \( \mathbb{F}^{n \times k} \) of what dimension?

15.5 Potpourri

**Exercise 15.13.** Let \( A \in \mathbb{Z}^{k \times n} \).

(a) Show: \( \text{rank}_{\mathbb{R}}(A) = \text{rank}_{\mathbb{Q}}(A) \).

(b) Let \( \text{rank}_{\mathbb{F}_p}(A) := \text{rank}_{\mathbb{F}_p}(A) \). Show: \( \text{rank}_{\mathbb{F}_p}(A) \leq \text{rank}_{\mathbb{R}}(A) \).

(c) Find a \((0,1)\)-matrix (every entry is zero or 1) such that \( \text{rank}_2(A) < \text{rank}_R(A) \).

Find short, elegant solutions.

**Exercise 15.14** (Pirate’s treasure). Show that not all the roots of \( x^{100} + 5x^{99} + 13x^{98} + \ldots \) are real. (The rest of the coefficients are unknown.)
Exercise 15.15. Let \( f = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \) with roots \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \). Assume \( a_0 a_n \neq 0 \). Let \( g = a_n + a_{n-1} x + \ldots + a_1 x^{n-1} + a_0 x^n \). What are the roots of \( g \)?

Exercise 15.16. Let \( f \in \mathbb{R}[x] \). If \( z = a + bi \in \mathbb{C} \) (\( i = \sqrt{-1} \)), the complex conjugate of \( z \) is \( \bar{z} = a - bi \). Let \( z \in \mathbb{C} \). Prove: if \( f(z) = 0 \) then \( f(\bar{z}) = 0 \); furthermore, \( \bar{z} \) and \( z \) have the same multiplicity.

Exercise 15.17. Let \( f \in \mathbb{R}[x] \). Then, \( f \) can be factored into linear and quadratic factors over \( \mathbb{R} \). (Of course, over \( \mathbb{C} \), \( f \) can be factored into linear factors.)

Exercise 15.18. Give a second proof of the statement: if \( n \) is odd and \( A \in M_n(\mathbb{R}) \), then \( A \) has a real eigenvector. (Use the preceding exercise.)

15.6 Matrices vs. Linear Maps

Let \( V \) be a vector space of dimension \( n \) and let \( W \) a vector space of dimension \( k \). Let \( \{e_1, \ldots, e_n\} \) be a basis of \( V \).

**Theorem.** \((\forall w_1, \ldots, w_n \in W)(\exists! \phi : V \to W)(\forall i)(\phi(e_i) = w_i).\)

We have seen the uniqueness of \( \phi \) in class.

**Exercise 15.19.** Prove the existence of \( \phi \) by defining \( \phi(x) = \sum \alpha_i w_i \) for any \( x = \sum \alpha_i e_i \in V \). (Check this gives a linear map and maps \( e_i \mapsto w_i \).)

We represent vectors as the column matrices by their coordinates with respect to the given basis. We write \( [x]_\xi = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{F}^n \) to express the relation \( x = \sum_{i=1}^n \alpha_i e_i \).

Next, we want to represent linear maps by matrices. In the light of the Theorem above, we need to state the image of each basis vector of \( V \). These images are in \( W \) so we need a basis of \( W \) for reference; let \( \{f_1, \ldots, f_n\} \) be a basis of \( W \). Let \( x \in V \).

Let \( \phi : V \to W \). We represent \( \phi \) by a matrix of which the \( j \)-th column lists the coordinates of \( \phi(e_j) \) in terms of the basis \( \{f_i\} \) of \( W \).

\[
[\phi]_{\xi,f} = \begin{bmatrix}
[\phi(e_1)]_{f} & \ldots & [\phi(e_n)]_{f}
\end{bmatrix}
\]

**Definition.** \( \text{Hom}(V,W) := \{\phi : V \to W : \phi \text{ is a linear map}\} \).

**Exercise 15.20.** As vector spaces, \( \text{Hom}(V,W) \cong \mathbb{F}^{k \times n} \), so \( \text{dim}(\text{Hom}(V,W)) = kn \). \( \cong \) denotes that there exists a bijection between the two spaces that is a vector space isomorphism.

**Exercise 15.21.** Let \( \phi \in \text{Hom}(V,W) \) and \( x \in V \). Show that coordinates of \( \phi(v) \) in terms of basis \( \{f_i\} \) of \( W \) can be calculated as

\[
[\phi(v)]_{f} = [\phi]_{\xi,f} \cdot [v]_{\xi}
\]

**Exercise 15.22.** Let \( A, B \in \mathbb{F}^{k \times n} \). If \( (\forall x \in \mathbb{F}^n)(Ax = Bx) \), then \( A = B \).
Exercise 15.23. Suppose that \( \phi \in \text{Hom}(V, W) \) and \( \psi \in \text{Hom}(W, Z) \). Suppose \( V, W, Z \) have bases \( e, f, g \) respectively. Show:

\[
[\psi \phi]_{e,g} = [\psi]_{f,g} \cdot [\phi]_{e,f}
\]

Note: this exercise shows why we multiply matrices the way we do.

Proof. We need only check that for any \( v \in V \), it holds that \( [\psi \phi][v] = [\psi][\phi][v] \). But, by repeated application of the previous exercise, \( [\psi](\phi[v]) = [\psi][\phi v] = ([\psi \phi])v \).

Exercise 15.24. Use the preceding exercise to give a straightforward proof, without any calculation, that multiplication of matrices is associative.

Example. Find the matrix of the linear transformation \( \frac{d}{dx} : \mathbb{R}^\leq_n x \to \mathbb{R}^\leq_n x \) relative to the basis \( b = 1, x, x^2, \ldots, x^n \). Then,

\[
\left[ \frac{d}{dx} \right]_{b,b} = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & 0 & 0 & \cdots & 0 \\ a_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix}
\]

Example. Let \( R_\theta \) denote the rotation of the plane by \( \theta \) about the origin. Let \( e_1 \) and \( e_2 \) be two perpendicular unit vectors in counter-clockwise order; let \( e = (e_1, e_2) \).

Then \( R_\theta(e_1) = \cos \theta \cdot e_1 + \sin \theta \cdot e_2 \) and \( R_\theta(e_2) = -\sin \theta \cdot e_1 + \cos \theta \cdot e_2 \). So

\[
[R_\theta]_e = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}
\]

Since \( R_\alpha \cdot R_\beta = R_{\alpha + \beta} \), we obtain

\[
\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \cdot \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}
\].

Infer the addition formulas for trig functions.

Exercise 15.25. Consider the basis \( f = \{f_1, f_2 = R_\theta(f_1)\} \). What is \([R_\theta]_f \)? Compute Tr and det of this matrix with the corresponding data of the Example above.