Exercise 22.1. Let $b$ be an orthonormal basis in the $n$-dimensional Euclidean space $V$ and let $\phi \in \text{Hom}(V, V)$. Prove that $\phi$ is a symmetric transformation iff $[\phi]_b$ is a symmetric matrix. (Recall that $\phi$ is a symmetric transformation if $(\forall v, w \in V)((\langle \phi v, w \rangle = \langle v, \phi w \rangle)$.

Exercise 22.2. Let $V$ be a Euclidean space and $\phi$ and orthogonal transformation of $V$. Prove: if $U$ is a $\phi$-invariant subspace then so is $U^\perp$.

Exercise 22.3. Let $A \in \mathbb{R}^{k \times n}$. Prove: $\text{rank}(A^T A) = \text{rank}(A)$.

Exercise 22.4. The adjoint of a matrix $A = (\alpha_{ij}) \in \mathbb{C}^{k \times n}$ is the matrix $A^* = \overline{A}^T$ (conjugate-transpose). So the $(i,j)$ entry of $A^* \in \mathbb{C}^{n \times k}$ is $\overline{\alpha_{ji}}$ where the bar indicates complex conjugate.

(a) What is $(AB)^*$?

(b) For $x \in \mathbb{C}^n$, calculate $x^* x$. Show that this number is always real. When is it zero?

Exercise 22.5. Let $A \in M_n(\mathbb{R})$ be a symmetric real matrix and let $\lambda \in \mathbb{C}$ be a complex eigenvalue of $A$. Show that $\lambda$ is in fact real. – Notice that this completes the (second) proof of the Spectral Theorem.

Exercise 22.6. Let $A$ be a block-diagonal matrix with diagonal blocks $M_i$. (So the $M_i$ are square matrices and so is $A$.)

(a) Express $\det(A)$ in terms of the determinants of the $M_i$.

(b) What are the eigenvalues of $A$ in terms of the eigenvalues of the $A_i$?

(c) What happens if $A$ is block-triangular?

Exercise 22.7. For every even value of $n$, find an orthogonal matrix $A \in O(n)$ such that $A$ has no real eigenvector.

Exercise 22.8 (Exercise 20.22). Find a curve in $\mathbb{R}^n$ in general position, i.e., any $n$ distinct points on the curve are linearly independent.

Exercise 22.10. (a) Show that $\det(AB) = \det A \cdot \det B$.
(b) Show that $\det(A^{-1}) = \frac{1}{\det A}$.
(c) Show that similar matrices have the same characteristic polynomial.
(d) Which integral matrices have integral inverses? (A matrix is integral if its entries are integers.)

Exercise 22.11. Find a totally isotropic subspace of dimension $\lfloor n/2 \rfloor$ in $\mathbb{F}_2^n$ and in $\mathbb{C}^n$.

Exercise 22.12. There are $m$ clubs in a town of $n$ citizens. Recall the Eventown rules:
(1) All clubs are distinct
(2) All clubs have an even number of members
(3) All pairs of clubs share an even number of members.
The “Eventown Theorem” states that under these rules, $m \leq 2^{\lfloor n/2 \rfloor}$.
(a) Prove the Eventown Theorem. Can you characterize maximal Eventown club systems in algebraic terms?
(b) With $n = 7$ citizens, find a system of clubs such that everyone is a member of a club. — Then do the same with any number $n \geq 7$ citizens.
(c) Prove: every maximal system of Eventown clubs is maximum.

Exercise 22.13. (a) Show that if $A \in M_n(\mathbb{Z})$, then $\text{rank}_\mathbb{R}(A) \geq \text{rank}_{\mathbb{F}_p}(A)$.
(b) Find a $3 \times 3$ $(0,1)$-matrix that is singular over $\mathbb{F}_2$ and non-singular over $\mathbb{R}$.

Exercise 22.14. Show that every monic polynomial with rational coefficients is the characteristic polynomial of a matrix with rational entries. (“Monic” means the lead coefficient is 1.) (Hint: companion matrix.)

Exercise 22.15 (Courant-Fischer). Let $A \in M_n(\mathbb{R})$ be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The Rayleigh quotient of $A$ is the function $R_A: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by
$$R_A(x) = \frac{x^T A x}{x^T x}.$$ 
Prove:
$$\lambda_k = \max_{\dim U=k} \min_{x \in U \setminus \{0\}} R_A(x).$$

Exercise 22.16 (Interlacing). Let $M$ be an $n \times n$ real symmetric matrix, and $M'$ the matrix obtained by removing the $i$th column and $i$th row of $M$ (note that $M'$ is symmetric). Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $M$, and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of $M'$. Prove that the eigenvalues of $M$ and $M'$ interlace, i.e., that we have
$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n.$$