24.1 Direct sum of subspaces

Exercise 24.1 (Exercise 22.8, 20.22). Solution to a curve in general position.

Let $S, T \subseteq V$. Define $S + T := \{ s + t : s \in S, t \in T \}$. Notice that if $U_1, U_2 \leq V$ then $U_1+U_2 \leq V$. Recall the Modular Equation: $\dim(U_1+U_2) = \dim(U_1)+\dim(U_2)−\dim(U_1 \cap U_2)$.

When $U_1 \cap U_2 = \{0\}$, we call the sum of $U_1$ and $U_2$ their direct sum. Notation: $U_1 \oplus U_2 = U_1 + U_2$ (meaning $U_1 \cap U_2 = \{0\}$).

Exercise 24.2. If $x \in U_1 \oplus U_2$, then $(\exists! u_1 \in U_1, u_2 \in U_2)(x = u_1 + u_2)$.

Theorem 24.3 (Direct Sum of Subspaces). Let $U_1, \ldots, U_k \leq V$. The following are equivalent:

1. $(\forall v \in V)(\exists! u_1 \in U_1 \ldots u_k \in U_k)(v = \sum_i u_i)$.
2. $V = \sum U_i$ and $\sum \dim U_i = \dim V$.
3. $V = \sum U_i$ and $(\forall i) \left( U_i \cap \left( \sum_{j : j \neq i} U_j \right) = \{0\} \right)$.

Exercise 24.4. Prove Theorem 24.3.

Exercise 24.5. Two subspaces $U, W$ are orthogonal if $(\forall u \in U, \forall w \in W)(u \perp w)$. Prove: the sum of pairwise orthogonal subspaces is a direct sum. Note: this exercise is true only over Euclidean spaces and does not hold over fields with respect to inner products that can have isotropic vectors.

Definition 24.6. Let $U \leq V, W \leq V$. $W$ is a complement of $U$ if $U \oplus W = V$, i.e., $U \cap W = \{0\}$ and $U + W = V$. In Euclidean space, $U^\perp$ is the orthogonal complement of $U$.

Exercise 24.7. In infinite dimensional Euclidean spaces, $U^\perp$ may not be a complement.

Exercise 24.8. Eigenvectors to distinct eigenvalues are linearly independent.

Exercise 24.9. Let $\phi : V \to V$. The eigensubspace for eigenvalue $\lambda$ is $U_\lambda := \{ v : \phi v = \lambda v \}$.
The sum of eigensubspaces is always a direct sum: \( \sum U_{\lambda_i} = \bigoplus U_{\lambda_i} \).

\( \phi \) has an eigenbasis \( \iff \sum \lambda U_{\lambda} = V \iff \bigoplus U_{\lambda} = V \iff \sum \dim U_{\lambda} = \dim V \).

**Exercise 24.10** (Right vs. left eigenvector). Let \( A \in M_n(\mathbb{F}) \). Suppose that \( x, y, \mu, \lambda \) satisfy \( \mu \neq \lambda, Ax = \lambda x, \) and \( y^T A = \mu y^T \). Show that \( y^T \cdot x = 0 \).

**Proof.** Notice that \( \mu y^T x = y^T Ax = \lambda y^T x \), so \( (\mu - \lambda)y^T x = 0 \). \( \square \)

**Exercise 24.11.** Suppose that \( A \in M_n(\mathbb{F}) \) is symmetric with eigenvalues \( \lambda \neq \mu \) with corresponding eigenvectors \( x, y \) (so that \( Ax = \lambda x \) and \( Ay = \mu y \)). Show that \( x \perp y \).

**Corollary 24.12.** If \( A \in M_n(\mathbb{R}) \) is symmetric, then \( \mathbb{R}^n \) is an orthogonal direct sum of the eigenspaces.

**Exercise 24.13** (Spectral Theorem enhanced). If \( A \in M_n(\mathbb{R}) \) is symmetric, then any list of orthonormal eigenvectors can be completed to an orthonormal eigenbasis.

### 24.2 Matrices vs. invariant subspaces.

**Similarity to triangular matrix**

Let \( \phi : V \rightarrow V \) be a linear map on \( V \). Let \( \underline{e} = (e_1 \ldots e_n) \) be a basis of \( V \). Recall that \([\phi]_{\underline{e}} = [[\phi(e_1)]_{\underline{e}} \cdots [\phi(e_n)]_{\underline{e}}] \).

**Exercise 24.14.** Let \( \phi : V \rightarrow V \) be a linear map on \( V \). Let \( \underline{e} = (e_1 \ldots e_n) \) be a basis of \( V \). Prove: \( e_1 \) is an eigenvector of \( \phi \) if and only if all entries of the first column of \([\phi]_{\underline{e}} \) below the diagonal are zero: \( ([\phi]_{\underline{e}})_{j1} = 0 \) for any \( j \geq 2 \).

**Exercise 24.15.** Let \( U_i = \text{span}(e_1 \ldots e_i) \). Then \( U_i \) is \( \phi \)-invariant \( \iff (\forall k = 1 \ldots i)(\phi(e_k) \in U_i) \iff ([\phi]_{\underline{e}})_{jk} = 0 \) for any \( k = 1 \ldots i \) and \( j = i + 1 \ldots n \) (the bottom left block is 0).

(Note: The \( k \)-th column of \([\phi]_{\underline{e}} \) is \([\phi(e_i)]_{\underline{e}} \), so we can read whether \( \phi(e_k) \in U_i \) based on the entries of the \( k \)-th column.)

**Exercise 24.16.** Each \( U_i \) is invariant \( \iff [\phi]_{\underline{e}} \) is upper triangular.

**Theorem 24.17.** *Over the complex numbers, every matrix is similar to an upper triangular matrix.*

**Corollary 24.18.** If \( V \) is an \( n \)-dimensional vector space over \( \mathbb{C} \) then there is a chain \( V = V_n > V_{n-1} > \cdots > V_0 = \{0\} \) of \( \phi \)-invariant subspaces, where \( \dim(V_i) = i \).

(Prove that the Corollary is equivalent to the Theorem.)

**Proof of Theorem 24.17.** By induction on \( n \). Pick an eigenvector \( e_1 \) of \( A \). Extend \( e_1 \) to a basis \( \underline{b} \) of \( \mathbb{C}^n \). Then, \( A \sim B \), where \( B = \begin{pmatrix} \lambda_1 & \cdots \\ 0 & C \end{pmatrix} \) is found by conjugating by the change of basis matrix to the basis \( \underline{b} \). \( B \) takes the claimed form since \( \text{span}(e_1) \) is \( B \)-invariant.
By the inductive hypothesis, \( C \) is similar to an upper triangular matrix, i.e., there exists \( S \in M_{n-1}(\mathbb{C}) \) such that \( S^{-1}CS \) is upper triangular.

Let now \( T = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S \end{pmatrix} \)

Claim: \( T^{-1}BT \) is upper triangular.

\[ 24.3 \quad \text{Polynomials of Matrices and Cayley-Hamilton Theorem} \]

Let \( f(x) = x^2 + 3 \). Then \( f(A) := A^2 + 3I \).

**Theorem 24.19.** Let \( A \in M_n(\mathbb{F}) \). Then there exists a polynomial \( f \in \mathbb{F}[x] \) with \( f \neq 0 \) such that \( f(A) = 0 \).

**Proof.** We need to show that there exists \( m \) such that \( I, A, A^2, \ldots, A^m \) are linearly dependent in \( M_n(\mathbb{F}) \). Since \( \dim M_n(\mathbb{F}) = n^2 \), so \( m = n^2 \) is sufficient.

We will show that in fact \( m = n \) is sufficient.

**Theorem 24.20 (Cayley-Hamilton Theorem).** Let \( A \in M_n(\mathbb{F}) \). Then \( f_A(A) = 0 \).

**Exercise 24.21.** Let \( g(x) \in \mathbb{F}[x] \) be a polynomial and let \( D \in M_n(\mathbb{F}) \) be the diagonal matrix

\[ D = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \]

Then \( g(D) = \begin{pmatrix} g(\lambda_1) \\ \vdots \\ g(\lambda_n) \end{pmatrix} \).

**Exercise 24.22.** Deduce the Cayley–Hamilton Theorem for diagonal matrices from the preceding exercise. (Recall: \( f_D(t) = \prod(t - \lambda_i) \).

**Exercise 24.23.** Let \( g \in \mathbb{F}[x] \). Show that \( g(S^{-1}AS) = S^{-1}g(A)S \).

**Exercise 24.24.** Deduce the Cayley–Hamilton Theorem for diagonalizable matrices from the preceding two exercises. (Recall: similar matrices have the same characteristic polynomial.)

We are now ready to finish the proof of Cayley-Hamilton over \( \mathbb{C} \) by a limit argument.

**Lemma 24.25.** Diagonalizable matrices are dense in \( M_n(\mathbb{C}) \). In other words, for any \( A \in M_n(\mathbb{C}) \), for all \( \epsilon > 0 \), there exists diagonalizable \( B \in M_n(\mathbb{C}) \) such that \( (\forall i,j)(|b_{ij} - a_{ij}| < \epsilon) \).

Let us say that a sequence of polynomials \( f_i \) of degree \( n \) converges to the polynomial \( g \) if for every \( j \), the coefficient of \( x^j \) in \( f_i \) converges to the coefficient of \( x^j \) in \( g \).

**Exercise 24.26.** Suppose that \( A_1 \ldots \rightarrow A \). Show that \( f_{A_1}, \ldots \rightarrow f_A \).

**Exercise 24.27.** Suppose polynomials \( f_1, f_2, \cdots \rightarrow g \) and matrices \( A_1, A_2, \cdots \rightarrow B \). Prove: \( f_1(A_1), f_2(A_2), \cdots \rightarrow g(B) \).
Proof of the Cayley–Hamilton Theorem over \( \mathbb{C} \). We need to show \( f_A(A) = 0 \). Consider a sequence of diagonalizable matrices, \( A_1, A_2 \ldots \rightarrow A \) that converge to \( A \). Then, by Exercise 24.26, \( f_{A_1}, f_{A_2}, \ldots \rightarrow f_A \). Furthermore by Exercise 24.27, we find that \( f_{A_1}(A_1), f_{A_2}(A_2), \ldots \rightarrow f_A(A) \). But the \( A_i \) are diagonalizable, so we already know that \( f_{A_i}(A_i) = 0 \), so \( 0, 0, \ldots \rightarrow f_A(A) \). Thus, \( f_A(A) = 0 \).

The Cayley-Hamilton Theorem follows for \( \mathbb{R} \) as \( \mathbb{R} \subseteq \mathbb{C} \). Furthermore, the Cayley-Hamilton Theorem follows for \( \mathbb{Z} \), so it follows also for \( \mathbb{F}_p \).

Exercise 24.28 (Identity principle). If \( f(x_1 \ldots x_n) \in \mathbb{C}[x_1 \ldots x_n] = 0 \) for all \( x_1 \ldots x_n \in \mathbb{C} \), then \( f \) is formally 0 (all coefficients cancel).

Exercise 24.29. Combine the Identity Principle and the fact that the Cayley-Hamilton Theorem holds over \( \mathbb{Z} \) to prove that it holds over all fields (in fact, over all commutative rings with identity).

Exercise 24.30. Let \( f(x) = x^4 + ax^3 + bx^2 + cx + 15 \), with \( a, b, c \in \mathbb{Z} \). If \( q \in \mathbb{Q} \) and \( f(q) = 0 \), then \( q \in \mathbb{Z} \) and \( q | 15 \).

### 24.4 Girth of graphs, adjacency matrix.

#### The Hoffman–Singleton Theorem

**Definition 24.31.** The *girth* of a graph is the length of the shortest cycle. The girth of a tree is \( \infty \).

**Exercise 24.32.** If \( G \) is regular of degree \( k \) with girth \( \geq 5 \), then \( n \geq k^2 + 1 \).

**Remark 24.33.** This bound is tight for \( k = 1, 2, 3, 7 \).

For \( k = 1 \), two vertices connected with one edge.

For \( k = 2 \), the pentagon.

For \( k = 3 \), Petersen’s graph.

For \( k = 7 \), \( n = 50 \), the Hoffman-Singleton graph.

**Exercise 24.34.** Show that the automorphisms group of the Petersen graph is isomorphic to \( S_5 \). Hint: \( 10 = \binom{5}{2} \).

**Theorem 24.35** (Hoffman-Singleton Theorem). Under the conditions that \( G \) is a \( k \)-regular graph with \( n \) vertices and girth \( \geq 5 \) satisfying \( n = k^2 + 1 \), it follows that \( k \in \{1, 2, 3, 7, 57\} \).

We prepare the proof with an exercise about the adjacency matrices of graphs.

**Recall:** the adjacency matrix \( A_G = (a_{ij}) \) of a graph \( G \) is defined by \( a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases} \).

In particular, \( A_G \) is a symmetric real matrix with zeros in the diagonal.

**Exercise 24.36.** Let \( B_G = (b_{ij})_{ij} := A_G^2 \). Then \( b_{ij} \) is the number of common neighbors of \( i \) and \( j \). In particular, \( b_{ii} = \deg(i) \).
Proof of the Hoffman–Singleton Theorem.

Exercise 24.37. Let $G = (V, E)$ satisfy the conditions of the HS theorem. If $x, y \in V$ and $x \sim y$, then $x, y$ have no common neighbor. Furthermore, if $x \neq y$ and $x \not\sim y$, then $x, y$ must have exactly one common neighbor.

So by Ex. 24.36, $B_G = A_G^2$ has entries $b_{ij} = \# \text{ common neighbors of } i, j = \begin{cases} k & i = j \\ 0 & i \sim j \\ 1 & i \not\sim j \end{cases}$.

Thus, $A_G^2 = kI + A_{\overline{G}} = k \cdot I + (J - A - I)$, where $J$ is the all-ones matrix and $\overline{G}$ is the complement of $G$. Thus,

$$A^2 + A - (k - 1)I = J. \tag{1}$$

Exercise 24.38. For any graph $G$ we have $A_G \mathbf{\bar{1}} = (\deg(1), \ldots, \deg(n))^T$, where $\mathbf{\bar{1}}$ is the all-ones vector $(1, 1, \ldots, 1)^T$.

Exercise 24.39. The graph $G$ is regular of degree $k$ if and only if $\mathbf{\bar{1}}$ is an eigenvector to eigenvalue $k$.

Returning to the proof of the HS Theorem, we observe by Exercise 24.39 that $\mathbf{\bar{1}}$ is an eigenvector of $A$ with eigenvalue $k$. Let us normalize $\mathbf{\bar{1}}$: let $e_1 = (1/\sqrt{n})\mathbf{\bar{1}}$. So $Ae_1 = k \cdot e_1$. Extend $e_1$ to an ON eigenbasis $e_1 \ldots e_n$ of $A$. It follows that for $i \geq 2$ we have $e_1 \perp e_i$ and therefore $\mathbf{\bar{1}}^T e_i = 0$.

Fix $i \geq 2$. Then, $Je_i = (\mathbf{\bar{1}}^T e_i, \ldots, \mathbf{\bar{1}}^T e_i)^T = 0$. Furthermore, $Ae_i = \lambda_i e_i$ and $A^2 e_i = \lambda_i^2 e_i$.

So $(A^2 + A - (k - 1)I)e_i = (\lambda_i^2 + \lambda_i - (k - 1))e_i$. On the other hand, this is equal to $Je_i = 0$, so for $i \geq 2$ we have

$$\lambda_i^2 + \lambda_i - (k - 1) = 0. \tag{2}$$

In other words, all eigenvalues $\lambda_i, i \geq 2$, are roots of the polynomial $t^2 + t - (k - 1)$.

Thus, in addition to $\lambda_1 = k$, there can only be 2 distinct eigenvalues.

Proof to be continued tomorrow.