5.1 Linear Algebra

Exercise 5.1. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of a matrix, show that
(a) $\text{tr}(M) = \sum_{i=1}^{n} \lambda_i$.
(b) $\det(M) = \prod_{i=1}^{n} \lambda_i$.

Exercise 5.2. Find the eigenvalues of
(a)
\[
M = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

(b)
\[
M = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}
\]

(c)
\[
M = \begin{pmatrix}
a & b & b & \cdots & b \\
b & a & b & \cdots & b \\
b & b & a & \cdots & b \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b & b & b & \cdots & a
\end{pmatrix}
\]

Exercise 5.3. Recall that a Hermitian matrix $M$ satisfies $M = M^\dagger$ (conjugate-transpose). Prove that the eigenvalues of a Hermitian matrix are real.

Exercise 5.4. Recall that the Rayleigh quotient $R_A(v)$ is defined by
\[
R_A(v) = \frac{\langle v, Av \rangle}{\langle v, v \rangle}.
\]
Let $A$ be an $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Prove $\max_{v \in \mathbb{R}^n \setminus \{0\}} R_A(v) = \lambda_1$, $\min_{v \in \mathbb{R}^n \setminus \{0\}} R_A(v) = \lambda_n$.

### 5.2 Graph Theory

**Exercise 5.5.** For any graph with vertices $V$ and edges $E$, prove that
\[ \sum_{v \in V} \deg(i) = 2|E|. \]

**Exercise 5.6.** Prove that the number of people on Earth who have made an odd number of handshakes is even.

**Definition.** Define the adjacency matrix $A_G$ of a graph $G$ by the rule that $A_{ij} = 1$ if $i \neq j$ and the vertices $i$ and $j$ are adjacent, and $A_{ij} = 0$ otherwise. (So $A$ is an $n \times n$ matrix with all entries 0 or 1.)

**Exercise 5.7.** Let $A_G$ denote the adjacency matrix of the graph $G$. Let $\mu_1 \geq \ldots \geq \mu_n$ be eigenvalues of $A_G$. Prove that
\[ \frac{1}{n} \sum_{i=1}^{n} \deg(u) \leq \mu_1 \leq \max_{i} \deg(i). \]

**Exercise 5.8.** With $A_G$ the adjacency matrix and $\mu_1$ the largest eigenvalue as above, prove that
\[ \mu_1 = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\langle v, Av \rangle}{\langle v, v \rangle}. \]

**Exercise 5.9.** Prove that
\[ \sum_{i=1}^{n} \mu_i^2 = 2|E| = \sum_{i \in V} \deg(i). \]

(Hint: think about $A^2$.)

**Definition.** Define the Laplacian matrix $L_G$ of a graph $G$ by the rule that $A_{ij} = -1$ if $i \neq j$ and the vertices $i$ and $j$ are adjacent, $A_{ij} = 0$ if $i \neq j$ and the vertices $i$ and $j$ are not adjacent, and $A_{ii} = \deg(i)$.

**Exercise 5.10.** For a graph $G$, let $L_G$ denote the graph Laplacian matrix. Prove that for all $v$,
\[ \langle v, L_G v \rangle = \sum_{i,j \in E} (v_i - v_j)^2. \]

**Exercise 5.11.** Prove that if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $L_G$, then $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$.

**Exercise 5.12.** Prove that the multiplicity of the eigenvalue 0 of $L_G$ is equal to the number of connected components of $G$.

**Exercise 5.13.** Compute eigenvalues and eigenvectors of the adjacency matrix $A_G$ and the Laplacian matrix $L_G$ for the following graphs:

(a) $K_n$, the complete graph on $n$ vertices

(b) $C_n$, the cycle of length $n$.  

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