Exercise 7.1. Prove that the following are number fields:

(a) \( \mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \} \)

(b) \( \mathbb{Q}[i] = \{ a + bi \mid a, b \in \mathbb{Q} \} \)

(c) \( \mathbb{Q}[\sqrt{n}] = \{ a + b\sqrt{n} \mid a, b \in \mathbb{Q} \} \)

for \( n \in \mathbb{Z}, \) \( n \) not a square. (\( i = \sqrt{-1}. \))

Exercise 7.2. (a) Prove that \( (3\sqrt[3]{2} \cdot 3\sqrt[3]{2}) \) cannot be written as \( a + b\sqrt[3]{2} \) for any \( a, b \in \mathbb{Q} \), and conclude that \( \{ a + b\sqrt[3]{2} \mid a, b \in \mathbb{Q} \} \) is not a number field.

(b) Prove that the set \( \mathbb{Q}[\sqrt[3]{2}] := \{ a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q} \} \) is a number field.

Exercise 7.3. Let \( U_1 \) and \( U_2 \) be subspaces of a vector space \( V \). When is \( U_1 \cup U_2 \) a subspace of \( V \)? Give a very simple necessary and sufficient condition in terms of a simple relation between between \( U_1 \) and \( U_2 \).

Exercise 7.4. Let \( V \) be a vector space over a number field. Prove that \( V \) is not the union of finitely many proper subspaces \( U \leq V \).

Exercise 7.5. Prove that the “Hilbert space”

\[ \ell_2(\mathbb{R}) = \{ (a_0, a_1, a_2, \ldots) \mid \sum_{i=0}^{\infty} a_i^2 < \infty \} \]

is a subspace of \( \mathbb{R}^\mathbb{N} \) (the space of sequences over \( \mathbb{R} \)).

Exercise 7.6. Let \( \mathbb{R}[x] \) denote the space of polynomials over \( \mathbb{R} \). Prove that the set \( \{ 1, x, x^2, \ldots \} \) is a basis for \( \mathbb{R}[x] \).

Exercise 7.7. Prove that the set \( \{ (1,0,0,\ldots),(0,1,0,\ldots),(0,0,1,\ldots),\ldots \} \) is not a basis for \( \ell_2 \).

Exercise 7.8. If \( S \) is a (finite or infinite) linearly independent subset of a vector space \( V \), then any subset of \( S \) is linearly independent.

Exercise 7.9. (a) Span(\( S \)) = \{ 0 \}
(b) For every subset $S \subseteq V$, $\text{Span}(S)$ is a subspace of $V$.

(c) $S \subseteq V$ is a subspace if and only if $S = \text{Span}(S)$. 
Exercise 7.10. “Continuum” is the cardinality of the set of real numbers. Prove that
(a) dim $\mathbb{R}^N = \text{“continuum”}$
(b) dim $\ell_2 = \text{“continuum”}$
(c) dim $\mathbb{Q} \mathbb{R} = \text{“continuum”}$ ($\mathbb{R}$ viewed as a vector space over $\mathbb{Q}$).

Exercise 7.11. Recall: a subset $S \subseteq \mathbb{Q}$ is a Dedekind cut if $S \neq \emptyset$, $S \neq \mathbb{Q}$, and $S$ is “downward closed, i.e., if $a, b \in \mathbb{Q}$, $a < b$, and $b \in S$ then $a \in S$.

(a) Prove that the Dedekind cuts form a chain.
(b) Define $\mathbb{R}$ as using Dedekind cuts.

Exercise 7.12. Recall that a basis of $V$ is a linearly independent set of generators. Let $S \subseteq V$.
Prove that $S$ is a basis of $V$ if and only if $S$ is a maximal linearly independent set.

Exercise 7.13. Prove that if $S$ is linearly independent and the list $S \cup \{v\}$ is not then $v \in \operatorname{Span}(S)$.

Exercise 7.14. (a) Prove that any chain in the poset of linearly independent sets in a vector space is bounded.
(b) Infer from Zorn’s lemma that every vector space has a basis.

Note: a basis of $\mathbb{R}$ over $\mathbb{Q}$ is called a Hamel basis.

Exercise 7.15. Prove that if $L \subseteq V$ is a linearly independent set then $L$ can be extended to a basis.

Exercise 7.16. Prove that if $S \subseteq V$ is a set of generators ($S$ spans $V$) then $S$ contains a basis.

Exercise 7.17. A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy the Cauchy equation if $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that:
(a) if $f$ satisfies the Cauchy equation and is continuous, then $f$ is linear.
(b) if $f$ satisfies the Cauchy equation and is continuous at a point, then $f$ is linear.
(c) if $f$ satisfies the Cauchy equation and is bounded in some interval, then $f$ is linear.
(d) if $f$ satisfies the Cauchy equation and is measurable, then $f$ is linear.

Exercise 7.18. Prove that there exist non-linear $f$ that satisfy the Cauchy equation. (Hint: Let $F = \mathbb{Q}[\sqrt{2}]$. Find a nonlinear solution of the Cauchy equation over $F$.)

Exercise 7.19 (Steinitz Exchange Principle). Let $L$ be a (finite or infinite) linearly independent set in $V$ and $W \subseteq V$. Assume $L \subseteq \operatorname{Span}(W)$. Prove: for any $v \in V$ there is a $w \in W$ such that the list $(V \setminus \{v\}) \cup \{w\}$ is linearly independent. (In particular, $w \notin V \setminus \{v\}$.)

Exercise 7.20 (First Miracle of Linear Algebra). Prove: if $v_1, \ldots, v_k \in V$ are linearly independent and all the $v_i$ belong to $\operatorname{Span}(w_1, \ldots, w_\ell)$ for some $w_i \in V$ then $k \leq \ell$. (Use the Exchange Principle.)

Exercise 7.21. Prove that if there exists a finite basis for $V$, then dim $V < \infty$. (Use the Exchange Principle.)
Exercise 7.22. Prove that if $B_1$ and $B_2$ are two bases for a finite-dimensional vector space $V$, then $|B_1| = |B_2|$. (Use the First Miracle — indeed this statement is equivalent to the first miracle.)