Day 1

Monday, June 19, 2017 9:30 AM

Solutions - elegant, clarity

↳ define things you use

↳ quantifiers

"Discovering Linear Algebra" - DLA

"Do" exercise - do it, do not hand in.

↳ collaborate if you don't understand

"HW" → due at beginning of next class - turn in.

Challenge -

Puzzle - AH-HA! → send notification if working on these.

Reward ♡

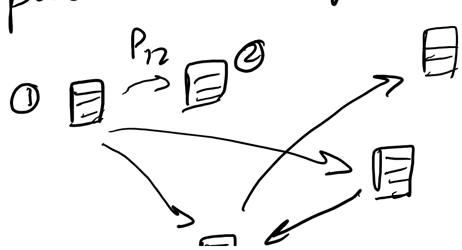
- Self-grading

Linear Algebra + Combinatorics + (Graph Theory)

Deck of cards - what does it mean to shuffle?

7 times to be "shuffled"?

52! possible arrangements.



"shuffling" - jumping from deck to deck

states of system - 52! of them.

transition probability - ① → ② is P_{12}

no change in probability w/ states

as # of moves increases, all states become approx. equally likely.

$T = (P_{ij})_{N \times N}$ $(N = 52!)$

$T^2 = \text{---}$ (2-step transition probability)

$\vdots$

Finite Markov Chains.

we can't do this empirically (too large...) -

need theory.

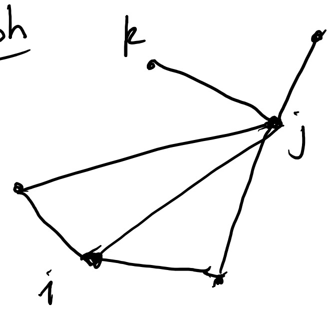
↳ Matrices - linear algebra

↳ Directed graph - combinatorics.

Largest eigenvalue of transition matrix - 1

$G = (V, E)$

Graph



$G = (V, E)$

V = set of vertices (nodes)

E = set of edges (links)

$i \sim j \rightarrow$ adjacency relation

$i \not\sim k$ (not adjacent)

Adjacency matrix $\quad V = [n] = \{1, \dots, n\}$

$n \times n$ matrix

$A_G = (a_{ij})$ where $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$

$i \not\sim i$ (not reflexive)

$i \sim j$ symmetric

not transitive

All diagonal elements are 0. $(a_{ii} = 0)$

A_G is symmetric matrix $(a_{ij} = a_{ji})$

degree $\quad \deg(i) = \#$ neighbors of i.

$\deg(j) = 5$

$\deg(i) = 3$

$\deg(i) = 3$

<u>Handshake Theorem</u>

$m = |E|$ (# edges)

$n = |V|$ (# vertices)

$$\sum_{i \in V} \deg(i) = 2m$$

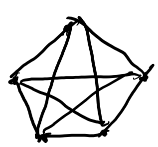
(each "handshake" counted twice)

$n = 13$

$(\forall i)(\deg i = 5)$?

→ total degree # is odd - not possible

Complete Graph K_n $\quad m = \frac{n(n-1)}{2} = \binom{n}{2}$



K_5

average degree $= \dfrac{\sum \deg(i)}{n} = \dfrac{2m}{n}$

$$\binom{n}{3} = \frac{n!}{(n-3)!\,3!} = \frac{n(n-1)(n-2)}{6}$$

Complement $\bar{G}$ of $G = (V, E)$

has $\bar{G} = \left(V, \binom{V}{2} \setminus E\right)$

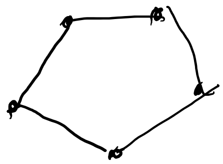
S - set

$\binom{S}{2}$ - set of pairs of elements of S.

$\left|\binom{S}{2}\right| = \binom{|S|}{2}$

Cycle of length n (≥ 3) - C_n

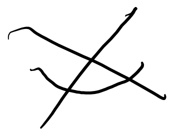
Cycle of length n (≥ 3) $\quad C_n$ $\qquad \left|\binom{V}{2}\right| = \binom{|V|}{2}$



$- \; C_5$

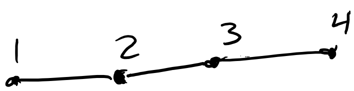
order = cardinality

$E(K_n) = \binom{V}{2}$



(self-loop)

Path of length $n-1$ = P_n (subscript - # of vertices in path)



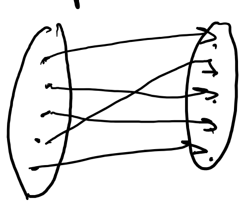
P_4

Isomorphisms

$G = (V, E)$ $\qquad H = (W, F)$

$f : V \rightarrow W$ bijection ↙ isomorphism.

s.t. preserves adjacency . . ?



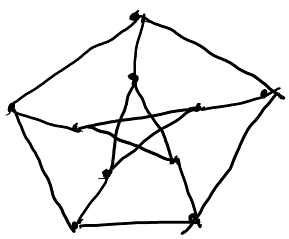
$(\forall i, j \in V)\ (i \sim_G j \Leftrightarrow f(i) \sim_H f(j))$

$G \cong H$ isomorphic if $\exists$ isomorphism $f : G \rightarrow H$

Graph Isomorphism Problem

How can we tell whether two graphs are isomorphic?

Petersen's Graph



Regular

G is regular of degree k if every vertex has degree k.

Two graphs that have same # of vertices and edges - not isomorphic.

(smallest)

Connected - able to get to anywhere

(minimally - connected graphs - remove any edge → becomes disconnected)

Subgraph

$H = (W, F) \subseteq G = (V, E)$

if $W \subseteq V$

$F \subseteq E$

G is connected if $\exists$ path between every pair of vertices.

G is a tree if

(a) connected

(b) no subgraph cycles.

vertices

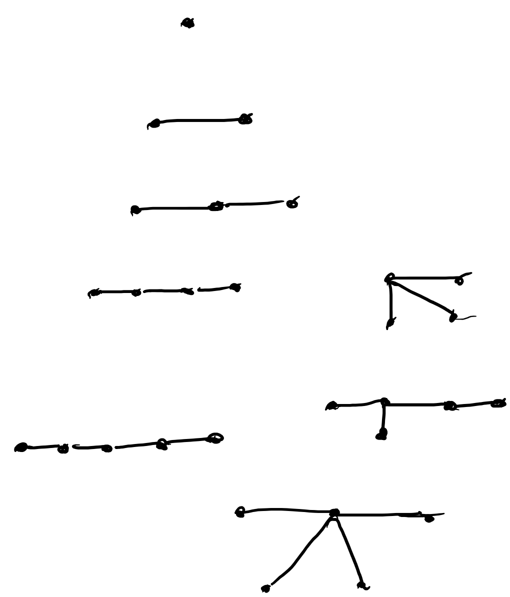
1

2

3

4

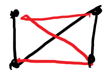
5

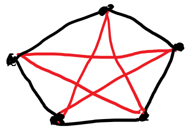


DO Find all trees with 6 vertices.

HW Same with 7 vertices. State # of non-isomorphic trees.

self-complementary : $G \cong \overline{G}$.

P_4  

C_5 

HW If $G \cong \overline{G}$ then $n \equiv 0$ or $1 \mod 4$

$a \equiv b \mod m$

"a is congruent to b modulo m"

means $m \mid a-b$

"m divides $a-b$"

2nd and 23rd fall on same day of week

$7 \mid 23-2 = 21$ ✓

b/c $2 \equiv 23 \pmod 7$

"calendar arithmetic"

DO* If $n \equiv 0$ or $1 \mod 4$ then $\exists G$ s.t. $G \cong \overline{G}$.

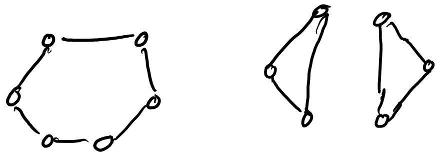
Why are ①  and ②  not isomorphic?

degree 3 vertex in ② but not in ①.

degree 2 vertex in ① but not in ②.

① has 2 degree 1 vertices but ② has 3.

If regular... cannot argue this way.



one contains 3-cycle, other does not

one is connected, other is not

DO $(\forall G)(G \text{ or } \bar{G} \text{ is connected})$.

G regular of degree k

$$\Downarrow$$

$\bar{G}$ regular of degree $n-k-1$

Ch All longest paths in a tree share a vertex.

(Not only pairs of paths)

(Paths cannot revisit vertices)

DO If T is a tree with ≥ 2 vertices $\exists$ vertex of degree 1

DO Use previous result to prove $m = n-1$ for a tree

DO T is a tree iff

$$(\forall i,j \in V)(\underset{\substack{\uparrow \\ \text{unique}}}{\exists !}\ i \cdots j \text{ path})$$

(DO) The following are equivalent for a graph G:

(1) G is a tree

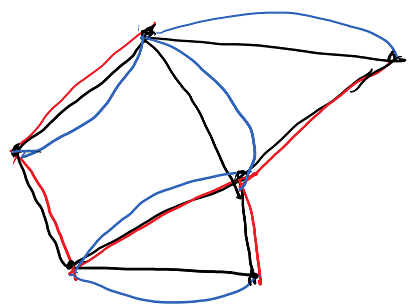
(2) G is connected and $m = n-1$

(3) G is cycle-free and $m = n-1$

Def. $H \subseteq G$ H is a spanning subgraph if $W = V$.

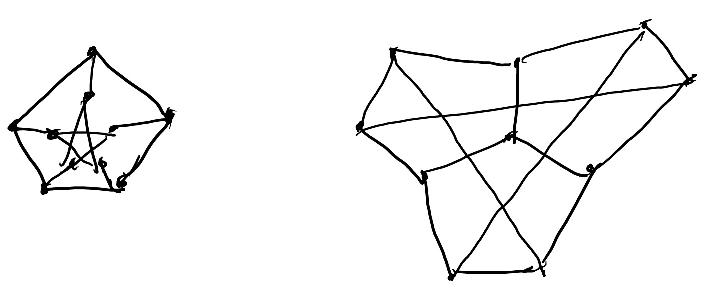
$H = (W, F)$

$G = (V, E)$



(DO) G has a spanning tree iff G is connected

Back to Petersen's graph...



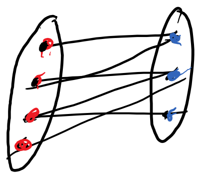
(HW) Are these isomorphic?

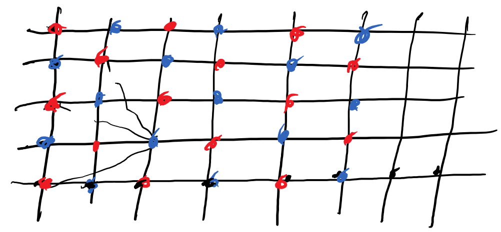
Bipartite graph

Bipartite graph

$V = V_1 \cup V_2$ (disjoint)

s.t. all edges go between the two parts





5

8

(all intersecting nodes)

$k \times \ell$ grid

Grid(k, ℓ)

$\boxed{\text{DO}}$ $n = k\ell$

$m = ?$

Bipartite?

$\rightarrow$ checkerboard coloring

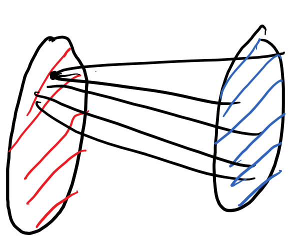
Associate graph w/ possible moves of knight

— bipartite as well.

15 moves — cannot end up on same space

(different color)

$\boxed{\text{DO}}$ G is bipartite $\Leftrightarrow$ G has no odd cycle



r s

$n = r+s$

$K_{r,s}$: complete bipartite graph

$n = r+s$

$m = rs$

(connect everything)

$n = r + s$ (connect everything)

$m \le rs$

Max # edges of bipartite graph w/ n vertices?

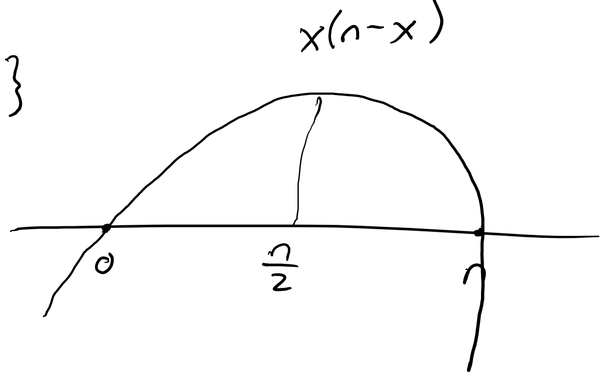
$$\max\{ r(n-r) \mid 0 \le r \le n \}$$

$$= \begin{cases} n^2/4 & \text{if } n \text{ even} \\ \frac{(n-1)}{2} \cdot \frac{(n+1)}{2} & \text{if } n \text{ odd} \end{cases}$$

$$= \frac{n^2-1}{4}$$

$$= \left\lfloor \frac{n^2}{4} \right\rfloor$$

floor



$\lfloor 5.3 \rfloor = 5$

$\lfloor -5.3 \rfloor = -6$

$\lfloor \pi \rfloor = 3$

$\lfloor 10 \rfloor = 10$

$C_5 \rightarrow$ not bipartites yet has no C_3

what is the max # of edges of a triangle - free graph?

$\boxed{\text{HW}}$ due Wednesday:

If $G \not\supset C_3$ (triangle - free)

then $m \le \frac{n^2}{4}$

$\boxed{\text{CH}}$ If $G \not\supset C_4$, then $\exists$ constant c s.t

$\boxed{\text{CH}}$ If $G \not\supseteq C_4$, then

$m < Cn^{3/2}$.

$\boxed{\text{CH}}$ This bound is tight. (with another C, for all sufficiently large n)

Determinants

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{\underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\text{determinant}}}$$

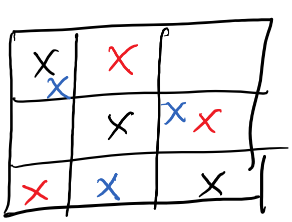
$$\left.\begin{array}{l} a_{11}x_1 + \cdots + a_{13}x_3 = b_1 \\ \vdots \\ a_{31}x_1 + \cdots + a_{33}x_3 = b_3 \end{array}\right\}$$

$$x_1 = \frac{- - -}{\begin{array}{l} a_{11}a_{22}a_{33} + \\ a_{12}a_{23}a_{31} + \\ a_{13}a_{21}a_{32} - \\ a_{11}a_{23}a_{32} - \\ a_{22}a_{13}a_{31} - \\ a_{33}a_{12}a_{21} \end{array}}$$

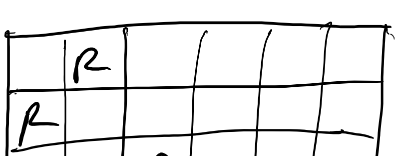
$2 = 2!$

$6 = 3!$

$24 = 4!$

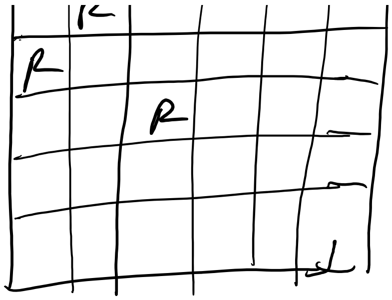


1 from every row and every column



$n!$ rook arrangements.

?



$n!$ rook

?

$A = (a_{ij})_{n \times n}$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$\det : \mathbb{R}^{n^2} \to \mathbb{R}$

$$\underline{\det A} = \sum_{\sigma \in \mathrm{Perm}_n} \underbrace{\pm a_{1,1\sigma}\, a_{2,2\sigma} \cdots a_{n,n\sigma}}_{\text{expansion terms}} \quad (n!)$$

$\sigma : [n] \to [n]$

permutation
(bijection of $[n]$
to itself)

$\sigma(2) : 2^\sigma$

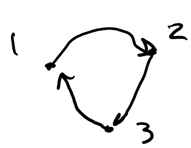
$(123) =$
$(231) = (312)$

$(23) = (32)$

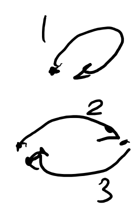
$a_{12}\, a_{23}\, a_{31}$

$a_{11}\, a_{23}\, a_{32}$

$1 \to 2$
$2 \to 3$
$3 \to 1$



$1 \to 1$
$2 \to 3$
$3 \to 2$



permutations

$\sigma, \tau \in \underbrace{\mathrm{Perm}_n}$

sigma tau

"symmetric group"

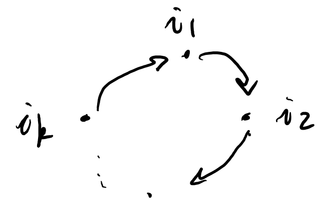
$i^{(\sigma\tau)} = (i^\sigma)^\tau$

(perform σ first, then τ)

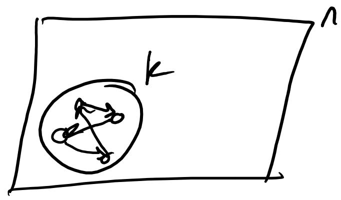
compositions.

sigma tau (symmetric group) (perform σ first, ...

compositions.



k-cycle



$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$\binom{n}{k}\cdot(k-1)! = \frac{n(n-1)\cdots(n-k+1)}{k}$$

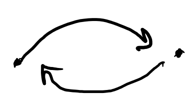
$k \geq 2$

(DO) Every permutation is a unique product of disjoint cycles.

unique: up to ordering

1-cycle: identity permutation

Def transposition: 2-cycle.



(DO) transpositions generate the entire symmetric group

$1 \ldots n$

$(i, i+1)$ – neighbor swap

(DO) Neighbor swaps generate the entire symmetric group

– how long does it take to get ... using transpositions?

from one state to another using . . .

DO How many steps do we need to generate the entire symmetric group?

In at most $n-1$ steps. (standard)

In order of n^2

$O(n^2) \to Cn^2$ suffices. $\to$ algorithm / procedure (example)

$\Omega(n^2) \to Cn^2$ is necessary. $\to$ proof. (more complicated. . .)

(neighbor swaps.)

DO* Product of even # of transpositions $\neq$ product of odd # of transpositions.

Def. σ is even if product of even # of transpositions

odd otherwise

DO For $n \geq 2$, # even perm. = # odd perm

$$\operatorname{sign}(\sigma) = \begin{cases} + & \text{if } \sigma \text{ even} \\ - & \text{if } \sigma \text{ odd} \end{cases}$$

"HW" Study* Ch. 6 in DLA ("Determinant")

* solve the exercises as well