6.34 (a) Find the determinant of

\[
\begin{vmatrix}
\alpha & \beta & \beta & \cdots & \beta \\
\beta & \alpha & \beta & \cdots & \beta \\
\beta & \beta & \alpha & \cdots & \beta \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta & \beta & \beta & \cdots & \alpha \\
\end{vmatrix}
\]

\[
\alpha (\alpha - \beta)^{n-1} + \beta \sum_{i=1}^{n-1} (-1)^i (\beta - \alpha)^{i} (\alpha - \beta)^{n-2-i}
\]

\[
\sum_{i=1}^{n-1} (-1)^i \left[ (-1)^i (\alpha - \beta)^{i} \right] (\alpha - \beta)^{n-2-i}
\]

\[
(n-1)(\alpha - \beta)^{n-1} \beta = (\alpha + (n-1) \beta)(\alpha - \beta)^{n-1}
\]

(b) \( A_n = \begin{bmatrix}
1 & 1 & 1 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
\end{bmatrix} \]

\( C_{11} = \det A_{n-1} \)

\( C_{21} = \det A_{n-2} \rightarrow \begin{bmatrix}
1 & 1 & 1 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{bmatrix} \)

\( \det (A_n) = \det (A_{n-1}) - \det A_{n-2} \)
\[ \begin{vmatrix} 1 & 1 & 1 \end{vmatrix} = 0 \]

\[ \begin{vmatrix} 1 & 1 & 0 \end{vmatrix} = -1 \]

\[ \begin{vmatrix} 1 & 1 & 1 \end{vmatrix} = 0 \]

\[ -1 \]

\[ 0 \]

\[ -1 \]

\[ 0 \]

\[ 0 \]

\[ 0 \]

\[ 0 \]

\[ \vdots \]

(C) same as above, but

\[ \det C_{11} = -\det A_{n-2} \]

and

\[ \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} = 2, \quad \text{so} \]

\[ \det A_n = \det A_{n-1} + \det A_{n-2} \]

1, 2, 3, 5, 8, 13, \ldots

6.3.5 Vandermonde.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \alpha_4 - \alpha_1 & \alpha_5 - \alpha_1 \\
0 & \alpha_2^2 - \alpha_2 \alpha_1 & \alpha_3^2 - \alpha_3 \alpha_1 & \alpha_4^2 - \alpha_4 \alpha_1 & \alpha_5^2 - \alpha_5 \alpha_1 \\
0 & 0 & 0 & 0 & 0 \\
0 & \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \alpha_4 - \alpha_1 & \alpha_5 - \alpha_1 \\
\end{bmatrix}
\]

multiply by \( \alpha_1 \) and subtract
\[ \det = \det \begin{vmatrix} \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \cdots & \alpha_n - \alpha_1 \\ \alpha_2 (\alpha_2 - \alpha_1) & \alpha_3 (\alpha_2 - \alpha_1) & \cdots & \alpha_n (\alpha_2 - \alpha_1) \\ \alpha_2^2 (\alpha_2 - \alpha_1) & \alpha_3^2 (\alpha_2 - \alpha_1) & \cdots & \alpha_n^2 (\alpha_2 - \alpha_1) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2^{n-2} (\alpha_2 - 1) & \alpha_3^{n-2} (\alpha_3 - 1) & \cdots & \alpha_n^{n-2} (\alpha_n - 1) \end{vmatrix} \]

\[ \prod_{j=2}^{n} (\alpha_j - \alpha_i) \]

\[ = \prod_{j<i} (\alpha_i - \alpha_j) \]

*If \( A \) is integer matrix, when is \( A^{-1} \) integer?*

*If \( \det A = \pm 1 \), \( \text{adj}(A) = (C_{ij})^T \)

\[ A^{-1} = \frac{\text{adj}(A)}{\det A} \]
\[ A = \det(A) \]

\[ A^{-1} \text{ must be integers since } \quad \adj(A) \text{ composed of cofactors (addition/multiplication)} \]

If \( A^{-1} \) is integer matrix, \[ \det A \det A^{-1} = \det I = 1 \]
\[ \det A \text{ and } \det A^{-1} \text{ must be integral, so } 1, -1, -1 \text{ are only pairs that are multiplicative inverses.} \]

If \( f: \mathbb{R}^m \rightarrow \mathbb{R}^n \) is linear, then \( \exists \)
\[ \tilde{a} \in \mathbb{R}^m \text{ s.t. } f(\vec{x}) = \tilde{a} \cdot \vec{x} \quad \forall \vec{x} \in \mathbb{R}^m \]

\[ \vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_m \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \]

\[ \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_m \end{bmatrix} \]

\[ f(\vec{x}) = f(x_1 \hat{e}_1 + x_2 \hat{e}_2 + \ldots + x_m \hat{e}_m) \]

\[ = f(x_1 \hat{e}_1) + f(x_2 \hat{e}_2) + \ldots + f(x_m \hat{e}_m) \]

\[ = x_1 f(\hat{e}_1) + x_2 f(\hat{e}_2) + \ldots + x_m f(\hat{e}_m) \]

let \( \tilde{a} = < f(\hat{e}_1), f(\hat{e}_2), \ldots, f(\hat{e}_m) > \).
\[ \text{sgn} (\sigma) = \begin{cases} 1 & \text{if \ even \ # \ of \ transpositions} \\ -1 & \text{if \ odd \ product \ of} \end{cases} \]

Transposition - swap value of two positions

\text{sgn} \ is \ well-defined \ (\text{Hard \ to \ show})

**WTS:** \( \text{sgn} (\sigma) = \text{sgn} (\sigma^{-1}) \)

\[
\begin{align*}
\sigma &\rightarrow 1 \rightarrow 2 \rightarrow 1 \\
&\rightarrow 3 \rightarrow 2 \\
&\rightarrow 1 \rightarrow 3
\end{align*}
\]

\[ \sigma = \tau_1 \tau_2 \cdots \tau_n \quad \sigma^{-1} = \tau_n \cdots \tau_2 \tau_1 \]

Transpositions are their own inverses

\[ (\tau_i)^2 = I \quad \forall i \in [n] \]

so \( \sigma \sigma^{-1} = I \).

\# of transpositions same, so same sgn. \( \square \)

If \( G \) is a triangle-free graph, then \( m \leq \frac{n^2}{4} \).

\( K_3 \notin G \) \quad \text{Induction?} \\
\[
\begin{align*}
\text{Induction?} & \quad \text{Induction?} \\
\text{n=1} & \quad \text{n=2} \\
\rightarrow & \quad \checkmark \\
\end{align*}
\]

\[
\begin{align*}
m &\leq \frac{n^2}{4} \\
m &\leq 1 \leq 1 \\
o &\rightarrow o
\end{align*}
\]
\[
\begin{align*}
    n \leq k-1 & \implies n \leq k \\
    \leq (k-2)+1 & = k-1 \\
    \leq \frac{(k-2)^2}{4} \\
    \leq \frac{k^2}{4} & \quad \Box \\
\end{align*}
\]

n even - complete bipartite graph works.

\[
\begin{align*}
    \deg(A) + \deg(B) & \leq n \\
    (n-2 \text{ odd} \text{ connected,} \\
    A \rightarrow B, \ B \rightarrow A) \\
\end{align*}
\]

\[
\begin{align*}
    n-2 \text{ odd} \text{ connected} & \\
\end{align*}
\]

\[
\begin{align*}
    \sum_{A \in V} \deg(A)^2 & = \sum_{E \in E} (\deg(A) + \deg(B)) \leq nm \\
\end{align*}
\]

Each vertex appears \(\deg(A)\) times in the sum.

\[
\begin{align*}
    \sum_{A \in V} \deg(A) & \text{ times} \\
\end{align*}
\]

HW: Finish this proof.

and we sum \((\deg(A))\)'s, so
\[
\sum_{A \in V} (\text{deg } A)^2 = \sum_{1 \leq A, B \leq E} (\text{deg}(A) + \text{deg}(B))
\]

Tree  Connected, acyclic graph

"forest"

A tree T with \( n \geq 2 \) has a vertex of degree 1 (leaf). for some \( i \in V \).

Can't have \( \text{deg}(i) = 0 \), if \( n \geq 2 \) - not connected.

If \( \exists i \in V \) s.t. \( \text{deg}(i) = 1 \),

\( \forall i \in V \) \( \text{deg}(i) \geq 2 \),

Pick a point and start walking - don't reuse edges. will eventually revisit a point (cycle) or terminate (vertex of deg 1?)

Maximum path - A path with the largest number of vertices.

Path - A path which cannot
to a large $1$

Every path between vertices of degree 1 is maximal. The converse is true for trees, not the other way around.

**Proof:** Let $T$ be a tree with $n \geq 2$. Take a maximal path $m \in T$.

(Since $T$ is connected and $n \geq 2$, $n \geq 1$.)

If the endpoints had deg $\geq 2$, it would have to connect to another pt. If not, it cannot be connected to any other pts in graph (must form a cycle), so must be separate. Then can extend path. Contradiction.

$\exists$ two vertices of degree 1.

---

... vertices has $n-1$ edges.
A tree with \( n \) vertices has \( n-1 \) edges.

- with 2 vertices, both have
  - \( \text{deg} \ 1 \)
  - 1 vertex \( \rightarrow 0 \)
  - 1 edge.

- \( n+1 \) vertex tree has at least 1 vertex
  of deg. 1.

remove it: (dis)

removed vertex cannot connect anything
(only 1 edge) and removing a vertex
cannot create a cycle

\( n \) vertex tree has \( n-1 \) edges by

inductive hyp

add back 1.

\( n+1 \) vertex tree has \( n \) edges. \( \Box \)

---

\( G \) is connected iff \( G \) has a spanning tree.

\( G \) is connected \( \Longleftrightarrow \) \( G \) has a spanning tree.

...original

connected \( \Rightarrow \) \( G \) has a spanning tree

connected \( \Rightarrow \) \( \text{added vertex} \)
Since connected, \( \exists \) path from every

to every other vertex.

Take union of all paths - remove
one cycle edge (redundant)

If \( G \) has \( 0 \) cycles, tree

If \( G \) has \( n \) cycles, it has a
spanning tree.

\( G \) has \( n-1 \) cycles, then if we remove
1 edge from a cycle, there will
be at least 1 free cycle
be at least 1 free cycle

Graph still connected (two ways to get
around \( 2 \rightarrow n \))

By strong induction, tree exists.

Two vertices are connected by a walk
iff they are connected by a path.

Maximal tree

Assume maximal tree is not
spanning \( \rightarrow \) contradiction.

Week 1 - Apprentice Page 10
A spanning tree $T$ of a connected graph $G$ is a tree that includes all vertices of $G$. If there exists a path between any two vertices of $T$, then $T$ is said to be maximal.

Extend the tree $T$ by adding an edge $ij$ between vertices $i$ and $j$ that intersects the path in the first edge on the path.

Thus, every maximal tree is spanning.

A maximal tree exists.

A spanning tree exists.

A graph is bipartite if it has no odd cycles.

Bipartite $\Rightarrow$ no odd cycles

If a graph contains an odd cycle, then it is not bipartite.
edge $A \to A$ or $B \to B$ -

No odd cycles $\Rightarrow$ bipartite.

Take a walk along the points, alternating colors - if conflict, you have an odd cycle.

Alternate color walk, every node has a unique path to each node.

Can have conflict edges.