Day 4

Thursday, June 22, 2017 9:35 AM

TA's office hours each lecture day

4-5 pm <u>Ry 162</u> "Theory Lounge"

today: Dylan

HW grading: $\frac{a}{15} + \frac{b}{35} \longrightarrow \frac{a+b}{50}$

Legal coloring of a graph $f: V \to \{\text{colors}\}$

s.t. if $i \sim j$ then $f(i) \neq f(j)$.

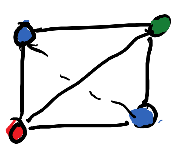
The **chromatic number** is the minimum # of colors required for a legal coloring

$$\chi(G) = \min\{k \mid G \text{ is } k\text{-colorable}\}$$

↑

\chi

K_4^-



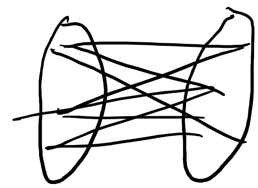
$\chi(K_4^-) = 3$

≤ 3 by exhibiting coloring

≥ 3 ?

Just because a vertex has high degree doesn't mean the chromatic number is high...

The complete bipartite graph with $\frac{n}{2}$ vertices on each side has a degree of $\frac{n}{2}$ for each point, but a chromatic number of 2



G is bipartite $\iff \chi(G) \leq 2$

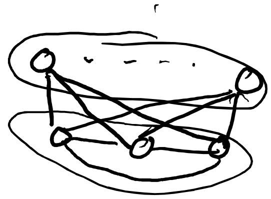
$\chi(G) = 1 \iff G = \overline{K_n}$ (empty graph)

<u>Def.</u> G is <u>k-colorable</u> if k colors suffice (i.e. $\chi(G) \leq k$).

$\chi(K_n) = n$ (all nodes connected)

$\chi(K_n^-) = n-1$ (complete graph minus an edge)

upper bound



1 color

n-2 vertices, all connected -

+ n-2 colors

n-1 colors

lower bound

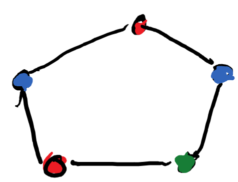
$K_n^- \supseteq K_{n-1} \rightarrow$ requires at least $n-1$ colors.

A <u>clique</u> (complete graph) of size n forces at least n colors.

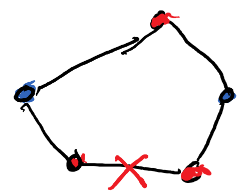
Paths and trees are bipartite - $\chi = 2$.

What about non-clique graphs?

C_5



≤ 3

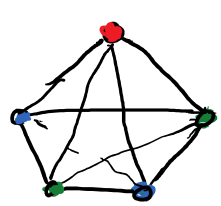


≥ 3

$$\chi(C_n) = \begin{cases} 2 & \text{if } \underbrace{2 \mid n}_{\text{divides}} \text{ (n is even)} \\ 3 & \text{otherwise} \end{cases}$$

Note bipartite $\iff$ no odd cycles and
bipartite $\iff$ 2-colorable.

Find $G \not\supseteq K_4$ s.t. $\chi(G) \geq 4$.



3-colorable.

$C_5 \supseteq K_3 \overset{?}{\Rightarrow} G \supseteq K_4$

$\chi(C_5) \geq 3 \qquad \chi(G) \geq 4$

DO

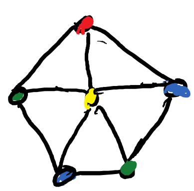
L : max clique size : $\leq q$

chromatic # : $\geq k$

$\hat{L}$: max clique size $\leq q+1$

chromatic # : $\geq k+1$

add new vertex connected to everything.



wheel W_6

(5 - spoke)

$\boxed{\text{HW}}$ (For Monday): Find a triangle-free graph that is <u>not</u> 3-colorable.

Hint: 11 vertices, 5-fold rotational symmetry

$\boxed{\text{CH}}$ $(\forall k)(\exists$ triangle-free graph G s.t. $\chi(G) > k)$

<u>Thm.</u> (Erdős, 1959)

$(\forall k, \ell)(\exists$ graph G of $\chi(G) > k$, without cycles of length $\leq \ell)$

$\boxed{\text{HW}}$ Max degree + 1 colors suffice for a graph. $(\chi \leq \deg_{max} + 1)$

(e.g. complete graphs)

The 4-color Theorem

Def G is planar if $\exists$ plane drawing of G (no intersections other than the vertices).

Thm Every planar graph can be colored by 4 colors.

Proof: If computer halts, theorem proved. If not, no.

Took 2 weeks of computer time at U of I.

Lemma. Every planar graph has a vertex of degree ≤ 5.

planar $\Longleftrightarrow$ spherical surface

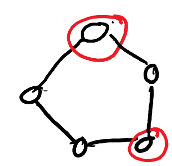
DO Find a famous planar graph, regular of degree 5.

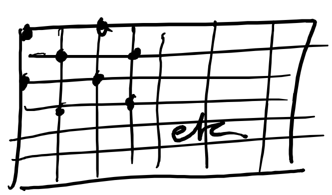
$\boxed{\text{HW}}$ (for Monday) Use the lemma to prove the 6-color theorem: Every planar graph is 6-colorable.

$A \subseteq V$ is <u>independent</u> if there is no edge among vertices in A.

$\alpha(G)$: size of largest independent set.

"independence number"





$n \times n$ grid

$\boxed{\text{HW}}$ (for Monday):

For $n \times n$ grid, α comes from checkerboard coloring.

(prove: upper bound)

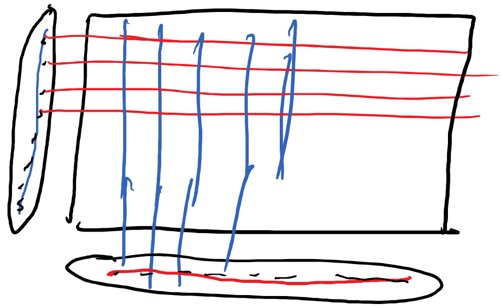
$\boxed{\text{HW}}$ If G is regular and non-empty, then $\alpha(G) \leq \frac{n}{2}$.

Def. Cartesian product of graphs:

$G = (V, E)$ $\quad K = G \square H = (U, D)$

$H = (W, F)$ $\quad U := V \times W$

Cartesian product of 2 paths: grid



Cartesian product of 2 cliques: possible rook movements

adjacency in K $\quad v_i \in V$, $w_i \in W$

$(v_1, w_1) \sim_K (v_2, w_2)$ if

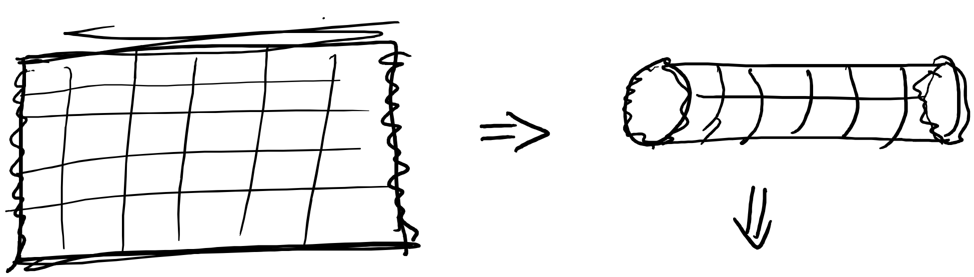
either $v_1 = v_2$ and $w_1 \sim_H w_2$

or $w_1 = w_2$ and $v_1 \sim_G v_2$

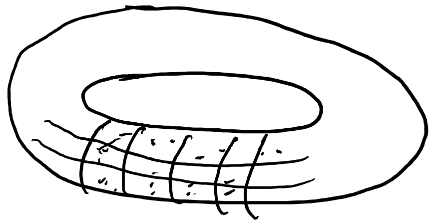
DO $\alpha(G \square H) \geq \alpha(G) \cdot \alpha(H)$

$\boxed{\text{HW}}$ Find $\alpha(C_5 \square C_5)$.

what does $C_n \square C_n$ look like?



If slope rational -
get back to
where you
started
(geodesics)
irrational - no



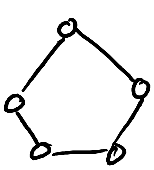
$\boxed{\text{HW}}$ $\alpha(G) \cdot \chi(G) \geq n$

DO Find graphs s.t

$\alpha(G) \cdot \chi(G) > cn^2$, where $c > 0$ is a constant.

(infinitely many graphs)

Def. An automorphism of $G = (V, E)$
is a $G \to G$ isomorphism.



5 rotations

5 reflections

$Aut(G) =$
automorphism group
(composition remains in group)

$|Aut(C_5)| = 10$

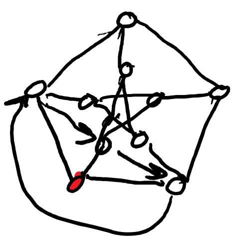
DO $|\mathrm{Aut}(\text{dodecahedron})| = 120$

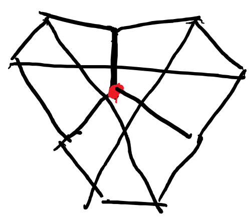
In 3-space
↓
orientation preserving vs. reversing.

$\boxed{\text{CH}}$ $\mathrm{Aut}(\text{dodecahedron}) \ncong S_5$ but

$\mathrm{Aut}^+(\text{dodecahedron}) \cong A_5$ (even permutations)

orientation preserving





Petersen graph

$\boxed{\text{CH}}$ $|\mathrm{Aut}(\text{Petersen})| = 120$

$\mathrm{Aut}(\text{Petersen}) \cong S_5$

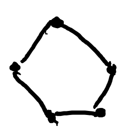
$G \leq \underbrace{\mathrm{Sym}(V)}$ all permutations of V

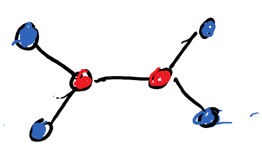
subgroup (closed under operations)

G is <u>transitive</u> if → 1 orbit (must be regular)

$(\forall x, y \in V)(\exists \sigma \in G)(x^\sigma = y)$

reflections and





2 orbits (isomorphisms preserve degree)

"detached rotations"

DO Find a regular graph with > 1 vertex s.t. $|Aut(G)| = 1$.

$\boxed{CH}$ If G is vertex-transitive then

$$\alpha(G) \cdot \chi(G) \leq n(1 + \ln n)$$

(discovered by Mario Szegedy)

<u>Def.</u> A <u>Hamilton cycle</u> is a cycle of length n that passes through all vertices.

G is <u>Hamiltonian</u> if $\exists$ Hamilton-cycle.

DO A dodecahedron is Hamiltonian.

DO Petersen's graph is <u>not</u> Hamiltonian

Is the torus vertex-transitive? Yes.
(rotate around each loop)

Is $k \times \ell$ grid Hamiltonian?

DO If k or ℓ is even, then $k \times \ell$ grid is Hamiltonian.

(there exist clear exceptions)

$\boxed{\text{HW}}$ If k and ℓ are odd, then $k \times \ell$ grid is <u>not</u> Hamiltonian. <u>AH-HA!</u>

(one-word answer $\rightarrow$ 4-word complete answer)

maximal independent set : cannot extend it

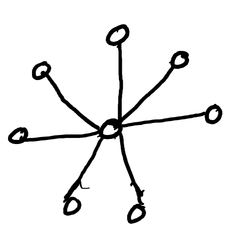
maximum independent set : largest possible

G has size

$\alpha(G)$

Find a graph where $\exists$ maxim<u>al</u> independent set that is very small compared to

$\alpha(G)$.



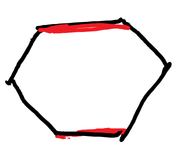
$K_{1,n-1}$ (star)

maximal : center

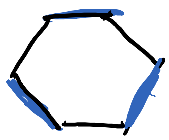
maximum : everything else

Ratio as high as $\frac{n-1}{1} = n-1$.

Matching in a graph is a set of disjoint edges.



maximal matching

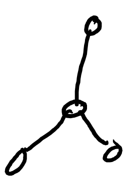


maximum matching

perfect matching: $\frac{n}{2}$ disjoint edges.
(hits all vertices)
(requires even # of vertices)

Empty graph - has even vertices, no matching

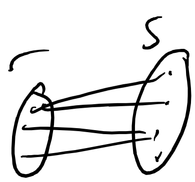
Find connected graph with even # vertices, no perfect matching.



$K_{1,3}$

$K_{r,s}$ has a perfect matching $\Leftrightarrow$ $r=s$

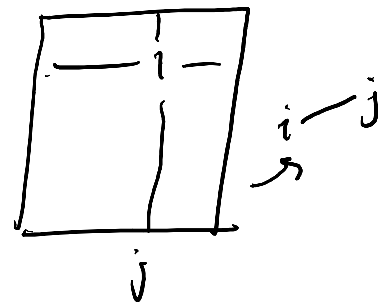
(all couples paired up)



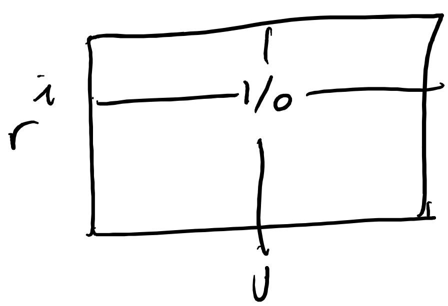
$G \subseteq K_{\frac{n}{2},\frac{n}{2}}$

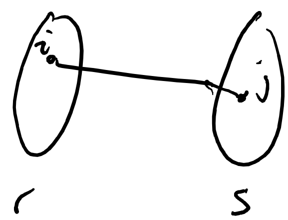
DO Find smallest connected bipartite graph with equal parts and no perfect matching.

Adjacency matrices.



Bipartite adjacency matrix





1 if adj.
0 if not

s

B_G

DO $G \subseteq K_{\frac{n}{2},\frac{n}{2}}$

B_G - bipartite adjacency matrix

$\det(B_G) \neq 0 \Rightarrow \exists$ perfect matching

$\boxed{\text{HW}}$ $\exists$ perfect matching $\not\Rightarrow \det(B_G) \neq 0$.

Question to contemplate:

Can we turn this idea into an efficient algorithm (as efficient as computing the determinant) to decide the existence of a perfect matching?

$\boxed{\text{CH}}$ Hadamard's Inequality

If $\underline{x} = (x_1, x_2, \dots, x_n)$

<u>norm</u> $= \|\underline{x}\| = \sqrt{\underline{x} \cdot \underline{x}} = \sqrt{\sum x_i^2}$.

$A \in M_n(\mathbb{R})$ $\qquad A = \begin{bmatrix} \underline{a}_1 \\ \text{rows} \\ \underline{a}_n \end{bmatrix}$

$\Rightarrow \quad |\det(A)| \leq \prod \|\underline{a}_i\|$.