Abstract vector spaces is something you can perform linear combinations on.

e.g., functions $\mathbb{R} \to \mathbb{R}$

$$\left\{ f : \mathbb{R} \to \mathbb{R} \right\} = \mathbb{R}^\infty$$

Subspace: $\mathcal{V} = [n] = \{1, \ldots, n\}$

$$\mathbb{R}^n = \mathbb{R}^n$$

$$\begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_n 
\end{pmatrix} = 
\begin{pmatrix}
\alpha(1) \\
\vdots \\
\alpha(n) 
\end{pmatrix}$$

$\mathbb{R}[t] = \text{polynomials over } \mathbb{R}$

$\mathbb{R}^\infty = \text{infinite sequences of real numbers}$

$$(a_0, a_1, \ldots)$$

$$(a(0), a(1), \ldots)$$ alternative function notation.
$\mathbb{R}$ "scalars"

$V$ vector space - elements of $V$ are "vectors".

Axioms

- $\forall (V, +)$ is a binary operation on $V$: $+: V \times V \to V$

1. $(V, +)$ is an abelian group.

   (a) $(\forall a, b \in V) (\exists! c \in V \text{ called } c = a + b)$ (addition is defined)

   (b) $(a + b) + c = a + (b + c)$ (addition is associative)

   (c) $(\exists 0)(a + 0 = 0 + a = a)$ (additive identity exists)

   (d) $(\forall a \in V)(\exists b)(a + b = b + a = 0)$ (additive inverse exists) notation: $b = -a$

   (e) $(\forall a, b \in V)(a + b = b + a)$ (addition is commutative)
2. **Multiplication by Scalars**

\[ \mathbb{R} \times \mathbb{V} \rightarrow \mathbb{V} \]

\[ (\lambda, a) \rightarrow \lambda a \]

(a) \((\forall \lambda \in \mathbb{R})(\forall a \in \mathbb{V})(\exists! \ b \in \mathbb{V} \text{ called } b = \lambda a)\)

(scalar multiplication is defined)

(b) \((\forall \lambda, \mu \in \mathbb{R})(\forall a \in \mathbb{V})(\lambda(\mu a) = (\lambda \mu) a)\)

(mixed associativity)

(c) \((\forall \lambda, \mu \in \mathbb{R})(\forall a \in \mathbb{V})(\lambda(a + \mu) = \lambda a + \lambda \mu)\)

(mixed distributivity - scalars)

(d) \((\forall \lambda \in \mathbb{R})(\forall a, b \in \mathbb{V})(\lambda(a + b) = \lambda a + \lambda b)\)

(distributivity - vectors)

3. \(1 \cdot a = a\) (rules out mapping everything to 0)

(axiom of normalization)
Cor. \((\forall x \in \mathbb{R})(\forall a \in V)(\lambda a = 0 \iff \lambda = 0 \land a = 0)\)

Proof:

(1) \(\lambda = 0 \Rightarrow \lambda a = 0\)

\(0 + 0 = 0\)

\((0 + 0)a = 0 \cdot a\) \hspace{1cm} (Multiply by \(a\))

\(0 \cdot a + 0 \cdot a = 0 \cdot a = f\) \hspace{1cm} (Axiom 2c)

\(f + f = f\)

DC: \(f = 0\).

\(f + f + (-f) = f + (-f)\) \hspace{1cm} (Axiom 1d - add \(-f\))

\(f = f + 0 = f + (f + (-f)) = (f + (f)) = 0\) \hspace{1cm} (Axioms 1b, 1c)

\(f = 0\)

\(\bigcirc\) If \(a = 0\) then \(\lambda a = 0\). (pot 2)
(3) If $\lambda \neq 0$ and $a \neq 0$ then $\lambda a \neq 0$.

\[
\frac{1}{\lambda} (\lambda a) = (\frac{1}{\lambda} \cdot \lambda) a = 1 \cdot a = a
\]

If $\lambda a = 0$

then $a = \frac{1}{\lambda} (\lambda a) = \frac{1}{\lambda} \cdot 0 = 0$ by (2)

so $a = 0$ (contradiction $\rightarrow a \neq 0$).

($\frac{1}{\lambda}$ exists b/c $\lambda \neq 0$).

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Linear combination of $a_1, \ldots, a_k \in V$:

\[
\sum_{i=1}^{k} \alpha_i a_i \quad \text{where } \alpha_i \in \mathbb{R},
\]

trivial l.c. - all $\alpha_i = 0$.

The list $a_1, \ldots, a_k$ is linearly independent

if only the trivial l.c. evaluates to 0.

if $S \subseteq V$ $\text{Span}(S) = \{ \text{all l.c. of finite subsets of } S \}$
Subspace: A vector space $W \subseteq V$ that is a vector space under the same operations.

All properties with universal quantifiers for operations in $V$ will be inherited by any subset of the space with the same operations.

**Thm.** $W \subseteq V$ is a subspace $\iff$

1. $0 \in W$.

2. If $a, b \in W$, then $a + b \in W$.

3. If $a \in W$, $\lambda \in \mathbb{F}$ then $\lambda a \in W$.

**Do** Span of any subset of $V$ is a subspace.

Note empty set is not a subspace.

Axiom 1c $\Rightarrow$ vector space is nonempty.
In particular, \( \text{Ov} = \text{Ov} \).

\[ \text{span(}\text{span}(S)) = \text{span}(S) \]

\text{Span} is an \underline{idempotent operator} - doing it twice is the same as doing it once (e.g. - projection for transformations.)

An \underline{involution} is when doing it twice is the same as doing nothing:

(Identity) \( \Rightarrow f(f(x)) = I(x) \).

\text{Rank} of a list \( L \) of vectors:

\[ \text{rk}(L) = \max \# \text{ of lin. independent vectors in } L. \]

\( \text{if no finite max, } \text{rank} = \infty \)

\text{Def. dim } V = \text{rk}(V).
Def. A basis of $V$ is a list $L$ s.t.
(1) $L$ is lin. independent
(2) $\text{Span}(L) = V$

Every linearly independent list of vectors is a basis of its span.

Thm. $L$ is a basis $\iff$ $L$ is a maximal lin. independent list.

(A) 

Lemma. If $v_1, \ldots, v_k$ are lin. indep. and $v_{k+1}, v_{k+2}, \ldots, v_k$ are lin. independent,
then $v_{k+1} \in \text{Span}(v_1, \ldots, v_k)$.

Proof. $(B) \Rightarrow \exists \alpha_1, \ldots, \alpha_k, \alpha_{k+1}$ not all zero

\[ \sum_{i=1}^{k+1} \alpha_i v_i = 0 \]

If $\alpha_{k+1} \neq 0$ then $v_{k+1} \in \text{Span}(v_1, \ldots, v_k)$ (move to other side, divide by $\alpha_{k+1}$).
So \( NTS: \alpha_{k+1} \neq 0 \).

Suppose \( \alpha_{k+1} = 0 \).

But then \( \sum_{i=1}^{k} \alpha_i v_i = 0 \) and not all of \( \alpha_1, \ldots, \alpha_k = 0 \) \( \implies \) contradiction (as \( v_1, \ldots, v_k \) are lin. indep.)

Proof of Thm.

basis \( \subseteq \) maximal lin. indep. list

(1) \( \implies \) Suppose \( v_1, \ldots, v_k \) basis

\( NTS: (\forall w \in V) (v_1, \ldots, v_k, w \text{ is lin. dep.}) \)

Since \( v_1, \ldots, v_k \) is basis,

\( \text{span} (v_1, \ldots, v_k) = V \), so

\( w \in \text{span} (v_1, \ldots, v_k) \) and

\( w \in \text{span} (v_1, \ldots, v_k) \implies w \text{ is lin. dependent} \)

\( \therefore v_1, \ldots, v_k \) maximal.
(2) Suppose \( v_1, \ldots, v_k \) is a maximal lin. indep. set.

\[ \text{NTS: } \text{Span}(v_1, \ldots, v_k) \subseteq V \]

i.e., \( \text{NTS: } (\forall w \in V), \ w \in \text{Span}(v_1, \ldots, v_k) \).

By maximality, \( v_1, \ldots, v_k \) are lin. independent,

so by lemma, \( w \in \text{Span}(v_1, \ldots, v_k) \).

Thus \( v_1, \ldots, v_k \) is a basis.

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Note: \( V \) is finite dimensional if all lin. indep. sets have bounded size, i.e.

\[ (\exists n_0)(\forall \text{ lin. indep. set})(|S| \leq n_0) \]

Thm. In a finite-dimensional space, every linearly independent list can be extended to a basis.

Cor. In a finite-dimensional space, \( \exists \) a basis.

Proof. Extend the empty list.
“Mathematics is about understanding the empty set”.

(Fermat’s Last Theorem)

Fermat’s Last Tango — musical

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**Isomorphism of Vector Spaces**

\[ f: \mathcal{V} \rightarrow \mathcal{W} \text{ bijection } \iff \]

1. \((\forall a, b \in \mathcal{V})(f(a + v_b) = f(a) + f(v_b))\)
2. \((\forall a \in \mathcal{V})(\forall \alpha \in \mathbb{R})(f(\alpha \cdot a) = \alpha \cdot f(a))\)

**00** If \( f \) is an isomorphism \( \mathcal{V} \rightarrow \mathcal{W} \) then

\( f \) is an isomorphism \( \mathcal{W} \rightarrow \mathcal{V} \).

\( f^{-1} \) is an isomorphism \( \mathcal{W} \rightarrow \mathcal{V} \).

\( \mathcal{V} \) is isomorphic to \( \mathcal{W} \) \( (\mathcal{V} \cong \mathcal{W}) \) if

\( \exists f: \mathcal{V} \rightarrow \mathcal{W} \text{ s.t. } f \text{ is isomorphism.} \)

**Thm.** \( \mathcal{V} \) is a basis of \( \mathcal{V} \) \( \iff \)

\[ (\forall w \in \mathcal{V})(\exists ! \alpha_1, \ldots, \alpha_k \in \mathbb{R})(w = \sum \alpha_i v_i) \]

coordinates of \( w \) w.r.t. the basis \( v_1, \ldots, v_k \).

(with respect to)
So \( B \) defines a map \( f: V \to \mathbb{R}^k \) coordination of \( V \)

\[
f(w) = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_k \end{bmatrix} \\
\left. \frac{\mathbf{w}}{} \right|_B
\]

\[
G_2 : \quad v_2 \quad w = 1.1v_1 + 3v_2 \\
\left. \frac{\mathbf{w}}{} \right|_B = \begin{bmatrix} 1.1 \\ 3 \end{bmatrix}
\]

\( B = (v_1, v_2) \)

**DO** \( \mathbf{w} \mapsto [\mathbf{w}]_B \) is an isomorphism.

**Cor.** If \( V \) has a basis consisting of \( k \) vectors then \( V \cong \mathbb{R}^k \).

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**First Miracle of Linear Algebra**

If \( v_1, \ldots, v_k \) are linearly independent and all \( v_i \in \text{Span}(v_1, \ldots, v_k) \) then \( k \leq 2 \).

"impossibility of boosting linear independence"
Cor. If $B_1, B_2$ are bases of $V$ then

$$|B_1| = |B_2|.$$ (i.e. maximal lin. indep. set is the maximum.)

**Proof.** Let $B_2$ be the $w_i$ and $B_1$ be the $v_j$. By First Miracle, $|B_2| \leq |B_1|$, by symmetry, $|B_1| \leq |B_2|$ and thus $|B_1| = |B_2|$.

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Back to extending lists for a second.

Then. In a finite dimensional space, every linearly independent list can be extended to a basis.

Finite dimensional - bounded # of lin. independent vectors.

Can always add more vectors that are lin. independent if not maximal - must stop at bounded $\Rightarrow$ maximal $\Rightarrow$ basis.
we actually don't need finite dimensionality.

Thm. Every vector space has a basis.

Proof. Use Zorn's Lemma.

REWARD PROBLEM

\[ f : \mathbb{R} \to \mathbb{R} \]

Cauchy's functional equation is satisfied:

\[ f(x + y) = f(x) + f(y) \]

(e.g., \( f(x) = c \cdot x \))

\[ f(x) = c \cdot x \]

(a) If \( f \) is continuous, then \( f \) is linear.

(b) If \( f \) is continuous at a point, then \( f \) is linear.

(c) If \( f \) is bounded on some interval, then \( f \) is linear.

\[ (c) \Rightarrow (b) \Rightarrow (a) \]
(d) If f is measurable then f is linear.
(e) I nonlinear solution.

(2) can be used for coefficients.

Prove: \(1, \sqrt{2}, \sqrt{3}\) are linearly independent over \(\mathbb{Q}\) (rationals).

i.e. if \(\alpha, \beta, \gamma \in \mathbb{Q}\) and \(\alpha(1) + \beta(\sqrt{2}) + \gamma(\sqrt{3}) = 0\),

then \(\alpha = \beta = \gamma = 0\).

\(\mathbb{R}\) is a vector space over \(\mathbb{Q}\).

HW connection:

For 2 pts., justify why the triangle-free graph with 11 vertices and 5-fold rotational symmetry is not 3-colorable.

Be sure to check grader's comments and take to friends \(\Rightarrow\) go to OH \(\Rightarrow\) speak to Prof.

Babai if something doesn't seem right.
From DLA:
15.1.11 (a)(b)
15.1.12
15.2.6 (a) - (c) Prove only "No" answers.

DO
15.3.11, 15.3.12, 15.3.22

First Miracle of Linear Algebra

Lemma. (Steinitz Exchange Principle)

Under conditions of First Miracle,
\((\forall i) (\exists j) (v_i, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k, w_j \in \text{indep.})\)

\(v_i \leftrightarrow w_j \quad \text{sharpen}\)

(Terminology: if \(S \subseteq V\) spans \(V\) then \(S\) is a set of \underline{generators}.)

Proof. Suppose \(w_j\) doesn't work. Then
\(v_i, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k, w_j \begin{array}{c} \text{in} \vphantom{\|} \text{indep.} \end{array}, \quad \text{but} \quad w_j \in \text{span}(v_i, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)
\)

\(w_j \in \text{indep.}, \quad \text{so} \quad w_j \in \text{span}(v_i, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k).
\)

by previous lemma.
If none of the $v_j$ work, then

$$w_1, \ldots, w_{k-1}, v_i, \ldots, w_k \in \text{Span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$$

but $v_i \in \text{Span}(w_1, \ldots, w_k)$ and

$$\text{Span}(w_1, \ldots, w_k) \subseteq \text{Span}(\text{Span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k))$$

$$\therefore v_i \in \text{Span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$$

$$\therefore v_1, \ldots, v_k \text{ are dependent - a contradiction.}$$

Thus, at least one of $v_j$ must work.

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**Proof of First Miracle**

Proof that $w_1, \ldots, w_k, v_i, \ldots, w_k$ is still in $\text{Span}(w_1, \ldots, w_k)$.

Still in $\text{Span}(w_1, \ldots, w_k)$,

all $v_j$ distinct (otherwise in dep. sublist)

in independent

$$k \leq l$$
Cor. If \( \mathbb{R}^n \cong \mathbb{R}^m \) then \( n = m \).

b/c \( \mathbb{R}^n \) has a basis of \( n \) vectors:

"standard basis" : columns of identity matrix

**Do** Show this is a basis.

**Do** If \( f : V \to W \) isomorphism, then \( f \) maps basis to basis.

Cor. \( \dim (\mathbb{R}^n) = n \).

Can find basis (n lin. indep.) and first miracle guarantees no longer

Def. \( A \in \mathbb{R}^{m \times n} \)

column rank \( (A) = \) rank of list of columns

row rank \( (A) = \) rank of list of rows

Second Miracle of Linear Algebra

\[
\text{col} \; \text{rk}(A) = \text{row} \; \text{rk}(A) =: \text{rk}(A) \quad \text{(rank of matrix A)}
\]
\[ A = [a_1, a_2, \ldots, a_k] \quad a_i : \text{columns.} \]

The **column space** of \( A \) is \( \text{Span}(a_1, a_2, \ldots, a_k) \).

**HW**
\[ \text{col } \text{rk}(A) = \text{dim of space}. \]

Proof: 1 line

**HW**
\[ A, B \in \mathbb{R}^{k \times k} \implies \]
\[ \text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B) \]

**CH**
Let \( A \) be matrix : \( A = (a_{ij}) \)
\[ B := (a_{ij}^2) \]

Prove: \( \text{rk}(B) \leq \frac{\text{rk}(A)(\text{rk}(A) + 1)}{2} \)

Let \( A \in \mathbb{R}^{k \times k} \).

\( B \) is a **right inverse** of \( A \) if \( AB = I_k \)

\( B \in \mathbb{R}^{k \times k} \), so \( AB \in \mathbb{R}^{k \times k} \).

**HW**
A right inverse exists \( \iff \text{rk}(A) = k \).

"A has full row rank."
Commentary:

A right inverse exists $\iff r_k(A) = k$ 
(from second miracle) $\iff$ rows are lin. indep. 
$\iff$ columns span $\mathbb{R}^k$

(Do) State analogous result for left inverse.

(HW) (for Monday) (submit via email)

Suppose we have

$A_1, \ldots, A_m, B_1, \ldots, B_m \in M_n(\mathbb{R})$

s.t. $A_i B_j = B_j A_i \iff i \neq j$.

Prove: $m \leq n^2$.

(Ask for a hint tomorrow - if you figure it out before tomorrow - email Prof. Babai)

Note: The chromatic polynomial problem is now due Mon. July 3 $\Rightarrow$ check the website for a more detailed description.