Monday July 3 deadline moves to Wed. July 5.

$\boxed{\text{HW}}$ (originally due yesterday)

If $v_1, \ldots, v_k \in \mathbb{R}^n$ are nonzero vectors, pairwise orthogonal then they are linearly independent.

(extended to tomorrow)

$\boxed{\text{HW}}$ $\left.\begin{array}{l} A \in \mathbb{R}^{k \times \ell} \\ B \in \mathbb{R}^{\ell \times m} \end{array}\right\} \operatorname{rk}(AB) \le \min\{\operatorname{rk}(A), \operatorname{rk}(B)\}$

$\operatorname{rk}(\underline{\underline{0}}) = 0$.

DO $\operatorname{rk}(A) = 0 \iff A = \underline{\underline{0}}$

$\boxed{\text{HW}}$ Find $A \in M_2(\mathbb{R})$ s.t. $A \neq \underline{\underline{0}}$ but $A^2 = \underline{\underline{0}}$.

DO If $A \in \mathbb{R}^{n \times k}$, $B \in M_k(\mathbb{R})$, and B is nonsingular, then $\operatorname{rk}(AB) = \operatorname{rk}(A)$.

DO Generalize to B not necessarily square, involving condition on $\operatorname{rk}(B)$.

$\boxed{\text{HW}}$ (for Wed July 5):

$A \in \mathbb{R}^{k \times l} \Rightarrow rk(A^T A) = rk(A)$

$\uparrow$ transpose

Thm. Apply elementary operation (i, j, λ) to $v_1, \dots, v_k \in V$.

$\longrightarrow v_1', \dots, v_k'$ where

$\boxed{\text{HW}}$ $v_l' = \begin{cases} v_l, & l \neq i \\ v_i - \lambda v_j & l = i \end{cases}$

Prove: $rk(v_1', \dots, v_k') = rk(v_1, \dots, v_k)$.

Proof: 1 line using a consequence of 1st miracle stated in class.

$\boxed{\text{DO!}}$ The column rank of a matrix does not change under elementary row operations.

(Do not use the Second Miracle.)

Proof: If columns are linearly independent $\Longleftrightarrow$ (operations reversible) they remain linearly independent after elementary row operation.

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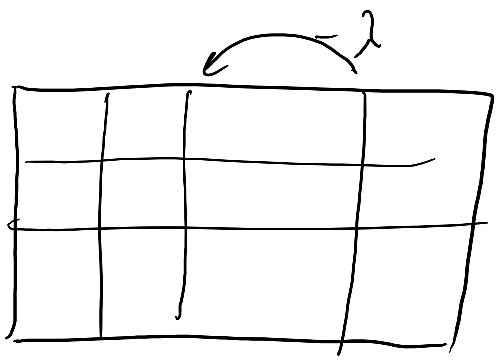
$r := \text{col rk}(A)$



r cols. lin indep.

$A \mapsto A'$

$\uparrow$

row op.

$\text{rk}(A') \geq \text{rk}(A)$ $\nearrow$ can get same set of lin. indep. vectors.

and

$\text{rk}(A) \geq \text{rk}(A')$ from reverse operation

so $\text{rk}(A) = \text{rk}(A')$. $\square$

Hint: $\text{rk}(A) = \dim$ col space

The same does not hold for elementary column operations.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{HW}}$ (for Wed. July 5)

If $G \not\supset K_3$ (triangle-free) then $\chi(G) = O(\sqrt{n})$,

big-oh

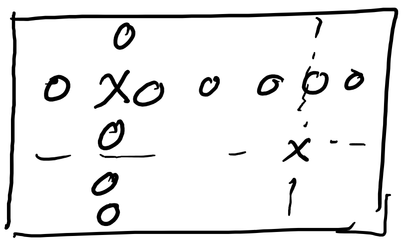
i.e. $(\exists c)(\forall$ sufficiently large $n \longrightarrow \chi(G) \leq c\sqrt{n}\,)$

estimate

implied constant.

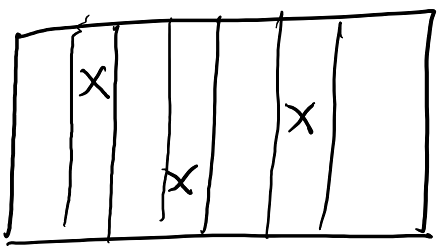
$\boxed{\text{DO!}}$ Use 1^{st} Miracle to show $rk(A) = \dim$ col. space.

Cor. Elementary row and column operations do not change either the row rank or the column rank



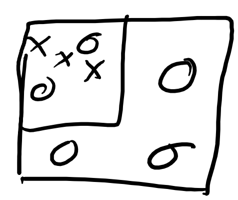
By elementary row + column ops, can eliminate everything else in a row and column

and result DO



In every row and column, there is at most 1 nonzero element

It is clear that each nonzero column/row is linearly independent, and



(permutations)

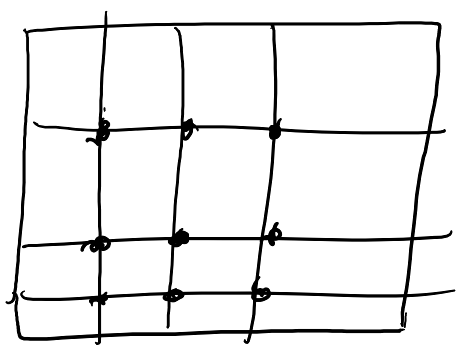
$\left.\begin{array}{l}\text{row}\\ \text{col}\end{array}\right\}$ rank = # non-zero entries

QED 2nd Miracle.

DO $(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

<u>Thm</u> $\text{rank}(A) = \max \{ t \mid \exists\ t \times t$ nonsingular submatrix in $A \}$



Review: For $A \in M_n(\mathbb{R})$, the following are equivalent:

(1) A is nonsingular, i.e., $\det A \neq 0$.

(2) The homogenous system of linear equations $Ax = 0$ has no nontrivial solution.

(3a.1) rows are linearly independent

(3b.1) columns are linearly independent.

(4) $\lambda = 0$ is not an eigenvalue.

(5a) $\exists A^{-1}$. (5b) $\exists$ left inverse (5c) $\exists$ right inverse.

(3a.2) has full row rank

(3b.2) has full column rank

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(6a) Rows span $\mathbb{R}^n$. (6b) Columns span $\mathbb{R}^n$.

(7a) Rows form a basis of $\mathbb{R}^n$.

(7b) Columns form a basis of $\mathbb{R}^n$.

Back to theorem:

$$\operatorname{rank}(A) = \max\{t \mid \exists\, t \times t \text{ nonsingular submatrix in } A\}$$

$\max$ ↑ m

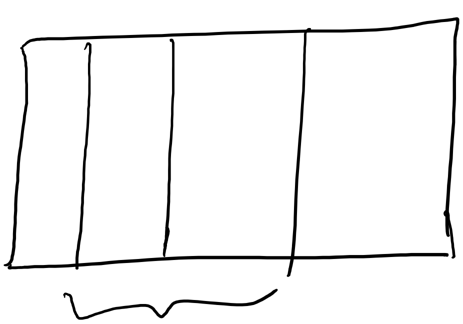
(1) $rk(A) \geq m$

NTS: If $\exists\, t \times t$ nonsingular submatrix,

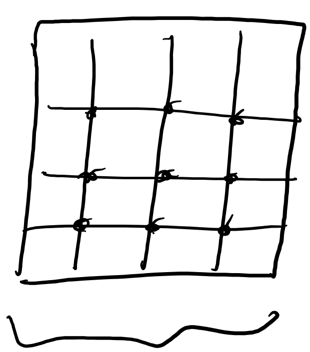
$rk(A) \geq t$. DO

(2) NTS: $rk(A) \leq m$

NTS: If $rk(A) \geq r$ then $\exists\, r \times r$ nonsingular submatrix.



lin indep.



r

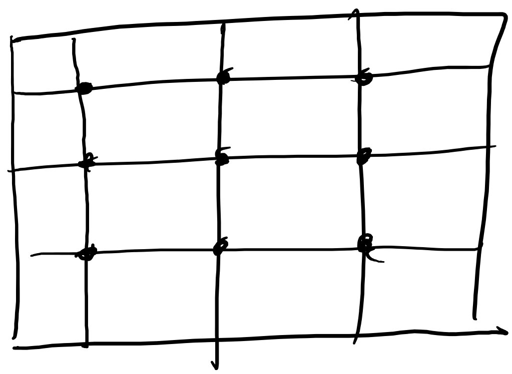
Need to find: r lin. indep. rows.

why does this suffice?

3a1 ⇒ 1.

Why does a set of r lin. indep rows exist?

By 2nd Miracle $\to$ col. rank r, so row rank $r \Rightarrow \exists\ r$ lin. indep. rows.



What about just taking r linearly independent rows and r linearly independent columns (by 2nd Miracle) and intersecting them?

DO Find a matrix A with s linearly independent rows and s linearly independent columns of which the intersection is all zero.

DO Can this happen to a matrix of rank s? (for, say, $s = 2$)

Vector space = "linear space"

$\varphi : V \to W$ is a <u>linear map</u> if

$(\forall x, y \in V)(\varphi(x+y) = \varphi(x) + \varphi(y))$

$(\forall x \in V)(\forall \lambda \in \mathbb{R})(\varphi(\lambda x) = \lambda \varphi(x))$.

phi

\varphi

\phi : ϕ

The <u>zero map</u> $\varphi = \underline{\underline{0}}$ defined by $\underline{\underline{0}}(x) = \underline{0}$ exists regardless of the spaces involved

$V = \mathbb{R}[t]$ → polynomials with real coefficients

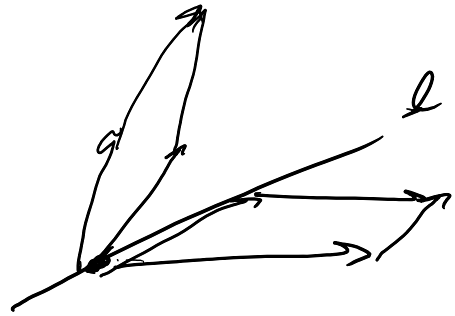
$D : f \to f'$ is a linear map.

$S : f \to \int_0^t f(x)\,dx$ is also a linear map.

$\mathbb{R}^{\mathbb{N}}$: left shift

space of sequences

If $W = V$: $\varphi : V \to V$ "linear transformation"

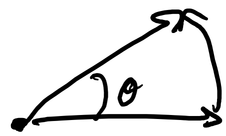


G_2 geom.

$\varphi : G_2 \to G_2$

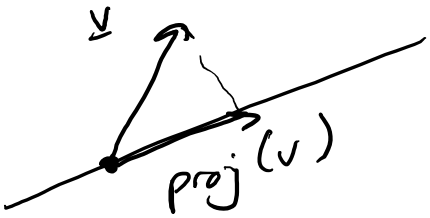
ρ_ℓ : reflection in line ℓ

rot_θ :



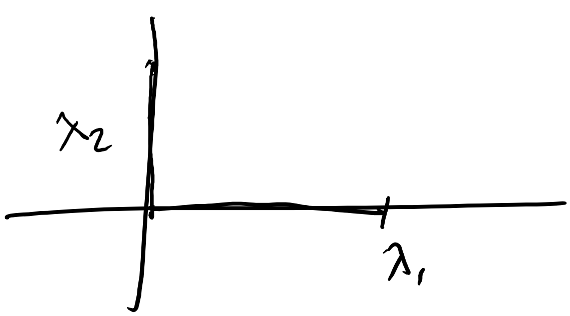
DO If $\varphi : V \to W$ then $\varphi(0v) = 0w$

proj$_\ell$



stretching : $\underline{v} \longmapsto \lambda \underline{v}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{pmatrix}$$



$Hom(V, W)$ denotes the set of $V \to W$ linear maps.

DO This is a subspace of W^V (of all $V \to W$ functions).

NTS: If $\varphi, \psi \in Hom(V, W)$ then $\varphi + \psi$ is also in it.

\psi

Also $\lambda\varphi \in Hom(V, W)$.

* Note:
linear independence not necessarily preserved

zero map, proj., derivative.

Question: what is dim $Hom(V, W)$?

Question: If $\varphi(v_1), \ldots, \varphi(v_k)$ are lin. indep., are $v_1, \ldots, v_k$ lin. indep.?

Eq. Contrapositive:

If $v_1, \ldots, v_k$ are lin dep., are $\varphi(v_1), \ldots, \varphi(v_k)$ lin dep.?

$\exists\, \alpha_1, \ldots, \alpha_k$ not all zero s.t. $\sum \alpha_i v_i = 0$.

$\Rightarrow \varphi(\sum a_i v_i) = \varphi(0) = 0$

and $\varphi(\sum a_i v_i) = \sum a_i \varphi(v_i)$ by properties of linear maps. □

(same coefficients are inherited.)

$\boxed{\text{DO !}}$

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Thm Let $v_1, \dots, v_n$ be a basis of V and $\underbrace{w_1, \dots, w_n}_{\text{any!}} \in W$. Then

$$(\exists!\, \varphi \in \mathrm{Hom}(V,W))(\forall i)(\varphi(v_i) = w_i)$$

"degrees of freedom" of φ?

$\dim V = n$ $\dim W = \ell$

in choosing
$$\left.\begin{array}{l} 1 : \ell \\ 2 : \ell \\ 3 : \ell \\ \vdots \\ n : \ell \end{array}\right\} \rightarrow = \boxed{n \cdot \ell}$$

total DoF

Thm. $\dim \mathrm{Hom}(V,W) = \dim V \cdot \dim W$

$\boxed{\text{CH}}$ Hilbert matrix

$2n$ distinct numbers

$\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$

$$H_n = \left(\frac{1}{\alpha_i - \beta_j}\right)_{n \times n} \quad (a_{ij} \text{ entry})$$

Prove:

(a) H_n is nonsingular.

(b) Find $\det H_n$ in factored form.

Please submit HW in LaTeX from now on

How do we tell the computer which linear map we are referring to?

$n \times k$ numbers ... arranged in a matrix?

"linear map" = "homomorphism of vector spaces"

dim $V = n$

dim $W = k$

Matrix associated with linear map

$\varphi : V \to W$

To coordinatize this, we need to pick a basis for V and W.

bases $\underbrace{e_1, \dots, e_n}_{= \underline{e}}$ for V and $\underbrace{f_1, \dots, f_n}_{= \underline{f}}$ for W

Def. $[\varphi]_{\underline{e}, \underline{f}} :=$

$$\Big[[\varphi(e_1)]_{\underline{f}} , [\varphi(e_2)]_{\underline{f}} , \dots , [\varphi(e_n)]_{\underline{f}} \Big]$$ $k \times n$ matrix.

If $w \in W$, $[w]_{\underline{f}} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_k \end{bmatrix}$ s.t. $w = \sum_{i=1}^{k} \gamma_i f_i$.

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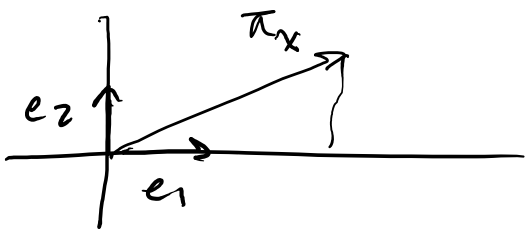
Look at projection



$$[\pi_x]_{\underline{e}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\pi_x(\underline{e_1}) = \underline{e_1}$

$\pi_x(\underline{e_2}) = \underline{0}$

$[e_1]_{\underline{e}} = a\underline{e_1} + b\underline{e_2}$

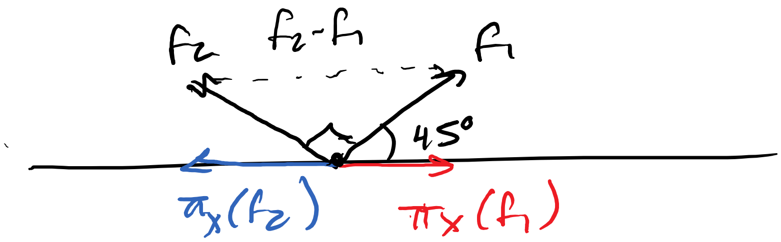
$a = 1, \quad b = 0$

If $\varphi : V \to V$,

$[\varphi]_{\underline{e}} = [\varphi]_{\underline{e}, \underline{e}}$

(choose same basis)

New basis: same transformation



$$[\pi_x]_{\underline{f}} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

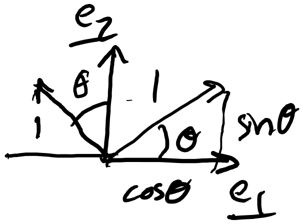
$\pi_x(f_1) = \frac{f_1}{2} - \frac{f_2}{2}$

$\pi_x(f_2) = -\pi_x(f_1) = \frac{f_2}{2} - \frac{f_1}{2}$

Notice the trace and determinant are the same between the two matrices.

Conjecture: changing the basis does not change the characteristic polynomial of the transformation

$[\text{rot}_\theta]_{\underline{e}}$



$\text{rot}_\theta(\underline{e_1}) = \cos\theta\,\underline{e_1} + \sin\theta\,\underline{e_2}$

$\text{rot}_\theta(\underline{e_2}) = -\sin\theta\,\underline{e_1} + \cos\theta\,\underline{e_2}$

DO

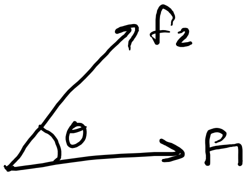
definition of cos and sin.

CCW rotation by θ

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\text{rot}_\theta(\underline{e_1})$ $\quad$ $\text{rot}_\theta(\underline{e_2})$

Let $\underline{f_1} = \underline{e_1}$, $\quad \underline{f_2} = \text{rot}_\theta(\underline{e_1})$ $\quad$ ($\theta \neq k\pi$ for $k \in \mathbb{Z}$)



$\boxed{\text{HW}}$ Find $[\text{rot}_\theta]_{\underline{f}}$.

(Compare trace and determinant from above -

trace $= 2\cos\theta$

det $= 1 \quad (\cos^2\theta + \sin^2\theta)$)

$\underline{x} \in V \quad$ basis $(e_1, \ldots, e_n) = \underline{e}$

$\varphi(\underline{x}) \in W \quad$ basis $(f_1, \ldots, f_k) = \underline{f}$

INPUT

$[\underline{x}]_{\underline{e}}$, $[\varphi]_{\underline{e},\underline{f}}$

OUTPUT

$k \times n \checkmark\ n \times 1$

$$[\varphi(\underline{x})]_{\underline{f}} = \boxed{[\varphi]_{\underline{e},\underline{f}} \cdot [\underline{x}]_{\underline{e}}}$$

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This justifies multiplication of matrices!

$\boxed{\text{HW}}$ Prove $[\varphi(x)]_{\underline{f}} = [\varphi]_{\underline{e},\underline{f}} \cdot [x]_{\underline{e}}$

$\boxed{\text{Do!}}$ $\underset{\underline{e}}{V} \xrightarrow{\varphi} \underset{\underline{f}}{W} \xrightarrow{\psi} \underset{\underline{g}}{X}$

$$(\psi\varphi)(x) = \psi(\varphi(x))$$

$$[\psi \cdot \varphi]_{\underline{e},\underline{g}} = [\psi]_{\underline{f},\underline{g}} \cdot [\varphi]_{\underline{e},\underline{f}}$$

$$[\text{rot } \alpha]_{\underline{e}} = R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_\alpha \cdot R_\beta = R_{\alpha+\beta}$$

$$[R_\alpha] \cdot [R_\beta] = [R_{\alpha+\beta}]$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Addition theorems for trig. functions. - special result of rotations in 2 dimensions.

$\mathbb{R}_k[t]$ = polynomials of degree $\leq k$

basis: $1, t, t^2, \ldots, t^k$.

$\boxed{HW}$ Find matrix of $f \mapsto f'$ in this basis.

$S = \text{Span}(\cos t, \sin t) \subseteq \mathbb{R}^{\mathbb{R}}$

(Right) $\text{shift}_\alpha := f(t) \mapsto f(t-\alpha)$ $\boxed{HW}$

Prove: this maps $S \to S$.

Find: $[\text{shift}_\alpha]_{\{\cos, \sin\}} = \begin{bmatrix} \vdots & \vdots \end{bmatrix}$

For tomorrow: 1/2 PS, 1/2 lecture.