Monday, July 10, 2017 9:33 AM

Problem Session

cyclotomic polynomials

n	primitive n^{th} roots of unity	Φ_n
1	1	$x-1$
2	-1	$x+1$
3	$-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$	x^2+x+1
4	$\pm i$	x^2+1
6	$\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$	x^2-x+1
p - prime		$\sum_{j=0}^{p-1} x^j = \frac{x^p - 1}{x-1}$

(0) $\deg(\gcd(f,f')) \geq 1 \iff$ multiple roots.

$\deg(\gcd(f,f')) \neq 0$

$\Rightarrow \exists p$

$$p \mid f, \quad p \mid f' \xrightarrow[\text{of Algebra}]{\text{Fund. Theorem}} (x-\alpha) \mid f, \quad (x-\alpha) \mid f'$$

$*(x-\alpha) \mid p$

$p \mid f' - pg'$

$f = pg$

$f' = pg' + p'g$

$p \mid pg'$

$(x-\alpha) \mid (x-\alpha)' g$

$\Rightarrow (x-\alpha) \mid g \Rightarrow (x-\alpha)^2 \mid f.$

Let $f = p^2 g$ $\quad$ p is the multiple root

$f' = 2pg + p^2 g'$

$= p(2g + pg')$

Since $p \mid f$ and $p \mid f'$, $\deg(\gcd(f, f')) \geq \deg(p) = 1$. $\square$

---

1. A diagonalizable $\Rightarrow \exists S, S^{-1}$ s.t. $B = S^{-1}AS$ where B is diagonal.

$SB = AS$

$\Rightarrow a_1, \ldots, a_n$ (col vectors of S) are eigenvectors for A

since S invertible, cols. are lin independent

A eigenbasis $v_1, \ldots, v_n$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad SD = \begin{pmatrix} \lambda_1 x_1 & & * \\ & \ddots & \\ * & & \lambda_n x_n \end{pmatrix}$$

$$S = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

$$AS = (Av_1 \cdots Av_n)$$
$$= (\lambda_1 v_1 \cdots \lambda v_n) \quad \text{by eigenvectors.}$$

$$D = S^{-1}AS \qquad \square$$

---

② $A \subset \mathbb{N}$, $|A| = k$

. $|A+A| \leq \binom{k+1}{2}$

$$A = \{a_1, \ldots, a_n\}$$
$$A+A = \{a+a' \mid a, a' \in A\}$$

$$|\{a_1 + a' \mid a' \in A\}| \leq k$$
$$|\{a_2 + a' \mid a' \in A\}| \leq k-1 \quad (a_2 + a_1 \text{ has already been counted})$$

$$\vdots$$

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} = \binom{k+1}{2}.$$

Monday, July 10, 2017 9:56 AM

$A + A = \{a_i + a_j \mid a_i, a_j \in A\}$

Case 1: $i = j$

n distinct sums

Case 2: $i \neq j$

counting: $\binom{n}{2}$ ways to pick

$$n + \binom{n}{2} = n + \frac{n(n-1)}{2}$$

$$= \frac{2n + n^2 - n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

What about in the general case?

$|\underbrace{A + A + \dots + A}_{\ell}|$

$a_{i_1} + a_{i_2} + \dots + a_{i_\ell}$ can be distinct for distinct combinations of i.

If not equal, we can assume strictly increasing.

Thus:

count $(i_1, i_2, \ldots, i_\ell)$ s.t.

$$1 \le i_1 \le i_2 \le \cdots \le i_\ell \le k$$

soln. "stars and bars" $\binom{k+\ell-1}{\ell}$

$$\overbrace{\star \,||\, \star \;|\; \star \quad \star \quad \cdots \quad \star}^{k \text{ stars}}$$

i_j = # of stars to left of bar j. ✓

$$\overbrace{\star \quad \star \quad \cdots \quad \star}^{k+\ell \text{ stars. ?}}$$

$$1 \le \underset{i_1}{2} \le \underset{i_2}{2} \le \underset{i_3}{3} \le 5$$

2 3 2 ?

3 2 2 ?

×

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\qquad$ $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ $\qquad$ $B^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ (Fibonacci)

$[\text{Shift}_\alpha]_{\underline{e}}$, $\quad \underline{e} = \{\sin t, \cos t\}$

$f \mapsto f(t-\alpha)$

$$\begin{pmatrix} \sin(t-\alpha) \\ \cos(t-\alpha) \end{pmatrix} = \begin{pmatrix} \sin(t)\cos(\alpha) - \sin(\alpha)\cos(t) \\ \cos(t)\cos(\alpha) + \sin(t)\sin(\alpha) \end{pmatrix}$$

$$= \sin(t)\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} + \cos(t)\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

image of basis vectors

$$\therefore \quad [\text{Shift}_\alpha]_{\underline{e}} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

DO 1. Find a non-singular matrix that is not diagonalizable.

2. Any strictly triangular matrix A is nilpotent; i.e. $\exists n \geq 1$ s.t. $A^n = \underline{0}$.

3. Find an example for $A \subset \mathbb{N}$, $|A| = k$, with $|A+A| \leq \binom{k+1}{2}$ where the bound is tight.

Monday, July 10, 2017 10:22 AM

Lecture:

$A \in M_n(\mathbb{C})$, with $\lambda \in \mathbb{C}$ an eigenvalue.

Is the set of all eigenvectors to λ a subspace?

$$\{\text{eigenvectors to eigenvalue } \lambda\} \overset{?}{\leq} \mathbb{C}^n$$

subspace?

No - $0 \notin$

What if we add 0?

$$\{0\} \cup \{\text{eigenvectors}\} = \{\underline{x} \in \mathbb{C} \mid \underbrace{A\underline{x} = \lambda\underline{x}}\}$$

homogenous system of linear equations.

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$(\lambda I - A)\underline{x} = \underline{0}$ has nontrivial solution

$\Updownarrow$

$\det(\lambda I - A) = 0$

$\Leftrightarrow f_A(\lambda) = 0$

$$\{\underline{x} \in \mathbb{C} \mid \underbrace{(\lambda I - A)}\underline{x} = \underline{0}\}$$

B - characteristic matrix of A.

Monday, July 10, 2017 10:30 AM

What is the dimension?

$U_\lambda(A)$ is the <u>eigensubspace</u> of A to eigenvalue λ.

$$\underbrace{\dim \{x \mid Bx = 0\}}_{\text{nullity}} = n - \overbrace{rk}^{rk}(B)$$

$x \in \mathbb{C}^n$

rank-nullity thm.

<u>$\dim U_\lambda$</u>

lin. indep. eigenvectors to eigenvalue λ.

<u>Def.</u> <u>geometric multiplicity</u> of eigenvalue λ.

$\dim U_\lambda = n - rk(\lambda I - A)$

λ is an eigenvalue $\iff rk(\lambda I - A) < n$

$\Updownarrow$

$\lambda I - A$ is singular.

$\therefore \geq 1 \iff \lambda$ eigenvalue

($\dim U_\lambda = 0$ says only $0 \in U_\lambda$ ... not an eigensubspace.)

	λ	geometric multiplicity
I:	1	n (choose any basis)

$$f_A(t) = (t-1)^n$$

$$\dim U_1(I) = n - rk(1 \cdot I - I)$$
$$= n - rk(0) = n \checkmark$$

$$B: \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 2 & & \\ & & & 2 & \\ 0 & & & & 2 \end{pmatrix}$$

λ	geometric multiplicity
1	2
2	3

$$f_B(t) = (t-1)^2 (t-2)^3$$

$$\text{geom mult}_B(2) = 5 - rk(\underbrace{2I - B}_{\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 0 & & \\ & & & 0 & \\ 0 & & & & 0 \end{pmatrix}})$$
$$= 5 - 2$$
$$= 3 \checkmark$$

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

λ	geom. mult.
1	1

$$f_C(t) = (t-1)^3$$

$$\text{geom. mult.}_C(1) = 3 - rk(\underbrace{1 \cdot I - C}_{\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}})$$
$$= 3 - 2$$
$$= 1$$

This is not diagonalizable, since there is only 1 lin. indep. eigenvector at most.

Over $\mathbb{C}$, $f_A(t) = (t-\alpha_1)^{k_1}(t-\alpha_2)^{k_2}\cdots(t-\alpha_r)^{k_r}$

	λ	geom. mult.	char. poly.	alg. mult.
I:	1	n	$(t-1)^n$	n
B:	1	2	$(t-1)^2(t-2)^3$	2
	2	3		3
			$(t-1)^3$	3
C:	1	1		

Note I, B diagonalizable and geom mult = alg. mult,

but C not and geom $\neq$ alg.

$\boxed{\text{HW}}$ For $A \in M_n(\mathbb{C})$,

(a) $(\forall \lambda)(\text{geom. mult}_A(\lambda) \leq \text{alg. mult}_A(\lambda))$

(b) A is diagonalizable $\Leftrightarrow$

$(\forall \lambda)(\text{geom. mult}_A(\lambda) = \text{alg. mult}_A(\lambda))$

Thm. $\forall$ field $\mathbb{F}$ $(\mathbb{C}, \mathbb{R}, \ldots)$, $\boxed{\text{DO!}}$

$\forall$ monic polynomial $f \in \mathbb{F}[t]$, $\deg f = n \geq 1$,

$\exists$ $n \times n$ matrix A s.t. $f_A = f$.

Elegant Solution: "companion matrix" (8.4.18 + 8.4.19 from DLA)

$$g(t) = (t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)$$
$$= a_0 + a_1 t + \ldots + a_{n-1}t^{n-1} + t^n$$

$$a_{n-1} = -\sum \alpha_i = -\sigma_1(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$a_{n-2} = \sum_{i<j} \alpha_i \alpha_j = \sigma_2(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$a_{n-3} = -\sum_{i<j<k} \alpha_i \alpha_j \alpha_k = -\sigma_3(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$\sigma_1, \sigma_2, \ldots, \sigma_n$ are the <u>elementary symmetric polynomials</u>.

$\vdots$

$$a_{n-j} = (-1)^j \sigma_j(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$a_0 = (-1)^n \prod \alpha_i = (-1)^n \sigma_n(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

Monday, July 10, 2017 10:54 AM

Let's look at the coeffs. of char. poly..

$$\begin{vmatrix} t-a_{11} & -a_{12} & - & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & - & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & & t-a_{nn} \end{vmatrix}$$

$$= b_0 + b_1 t + \cdots + b_{n-1} t^{n-1} + t^n$$

$$b_{n-1} = -Tr(A) = -(a_{11} + \cdots + a_{nn})$$

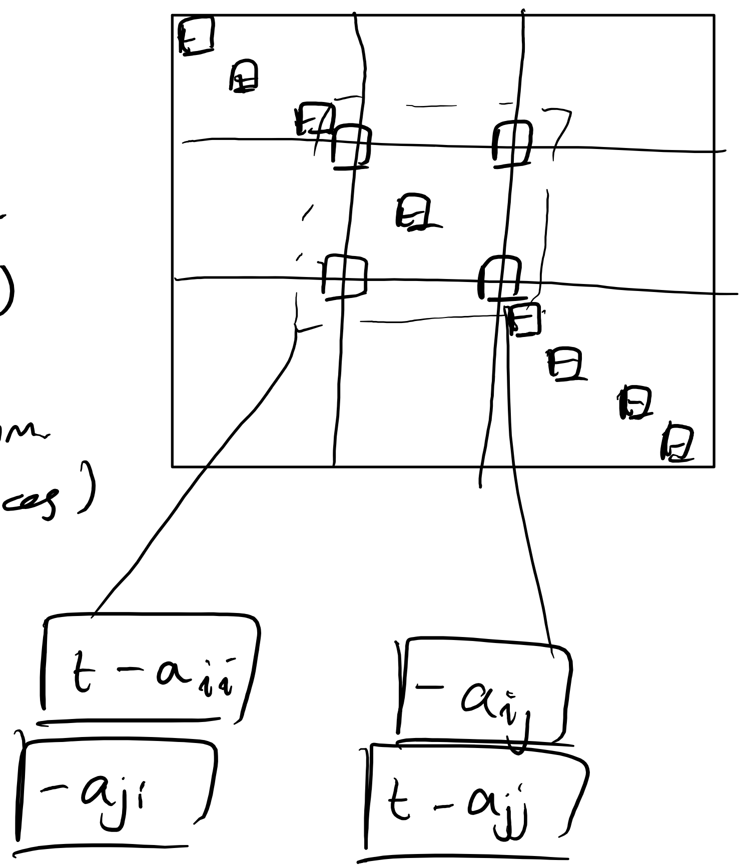
$$b_{n-2} = ?$$

$= \sum \det$ (2x2 symm submatrices)

$b_{n-3} = -\sum \det$ (3x3 symm submatrices)

$\vdots$

$b_{n-j} = (-1)^j \sum \det$ (jxj symm submatrices)



$$b_0 = (-1)^n \det(A).$$

$f_A(t) = b_0 + b_1 t + \dots + b_{n-1} t^{n-1} + t^n$

$= (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n)$

λ_i eigenvalues – listed with alg. mult (each alg. mult$_A(\lambda)$ times.)

<u>coeff t^{n-1}</u> $\quad -Tr(A) = -(\sum \lambda_i)$

<u>Thm.</u> $\sum \lambda_i = Tr(A)$.

$$\begin{pmatrix} 5 & & * \\ & 7 & \\ * & & -3 \end{pmatrix}$$

$\sum \lambda_i = 9$.

<u>constant</u>: $(-1)^n \det(A) = (-1)^n \prod \lambda_i$

<u>Thm.</u> $\prod \lambda_i = \det(A)$.

<u>Cor.</u> A is singular $\iff$ $\lambda = 0$ is an eigenvalue.

$\boxed{\text{HW}}$ If $A, B \in M_n(\mathbb{C})$, then

$\underbrace{AB - BA}_{\text{commutator}} \neq I$.

(<u>Cor.</u> Quantum mechanics has no finite-dimensional model.)

$\boxed{\text{HW}}$ $\mathbb{R}[t]$: space of real polynomials

$D := \frac{d}{dt}$ find $M: \mathbb{R}[t] \to \mathbb{R}[t]$

$D(f) := f'$

s.t. $MD - DM = I$.

$\boxed{M : \text{very simply defined}}$

Invariant subspace of linear transformation

$\varphi : V \to V$.

$W \leq V$ is an <u>invariant subspace</u> if

$\varphi(W) \subseteq W$; i.e., $(\forall w \in W)(\varphi(w) \in W)$.

- plane passing through origin $\perp$ to axis of rotation.

$$\mathbb{R}^{\leq n}[t] = \{ f \in \mathbb{R}[t] \mid \deg f \leq n \}$$

$$\dim \mathbb{R}^{\leq n}[t] = n+1$$

$\frac{d}{dt}$ operator — invariant subspaces?

— $\{0\}$, $\mathbb{R}^{\leq n}[t]$

$\mathbb{R}^{\leq k}[t]$: $0 \leq k \leq n$

$(\mathbb{R}^{\leq k}[t] \subseteq \mathbb{R}^{\leq n}[t])$

$\boxed{\text{HW}}$ (for wednesday): The $\mathbb{R}^{\leq k}[t]$ where $0 \leq k \leq n$ are the only $\frac{d}{dt}$-invariant subspaces.

$\boxed{\text{HW}}$ cyclic permutation matrix

$$\begin{tikzcd} e_1 \arrow[r, mapsto] & e_2 \arrow[r, mapsto] & \cdots \arrow[r, mapsto] & e_n \arrow[lll, bend left, mapsto] \end{tikzcd}$$

cyclically permute the basis vectors...

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Find eigenvalues and an eigenbasis over $\mathbb{C}$.

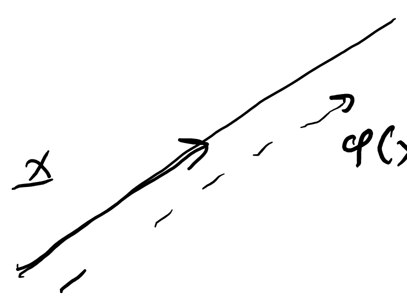
$\boxed{\text{CH}}$ 16.4.34(a): Find all invariant subspaces of cyclic permutation matrix.

__Def__ $A \sim B$ if $(\exists C$ nonsingular$)(A = C^{-1}BC)$
$(n \times n)$

DO This is an equivalence relation on $M_n(\mathbb{F})$, where $\mathbb{F}$ is any number field

__Thm__ Over $\mathbb{C}$, every $A \in M_n(\mathbb{C})$ is similar to a triangular matrix.

DO
1-dim invariant subspace = Span of an eigenvector.



$\varphi(x) = \lambda x$

$\boxed{\text{HW}}$ If $A, B \in M_n(\mathbb{F})$ and $AB = BA$, then $(\forall \lambda)(\underbrace{U_\lambda(A)}_{\text{eigensubspace}}$ is B-invariant$)$.

Suppose: $\varphi : V \to V$ lin. transformation.

$U \leq V$ $\qquad$ U is φ-invariant.)

$e_1, \ldots, e_k$: basis of U

extend to basis of V.

then U is φ-invariant $\Leftrightarrow$

$$[\varphi]_{\underline{e}} = \begin{array}{cc} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \left[\begin{array}{c|c} & \\ \hline 0 & \end{array}\right] \end{array}$$

this block is 0.

If U invariant,

$\varphi(e_1) \in U$,

so

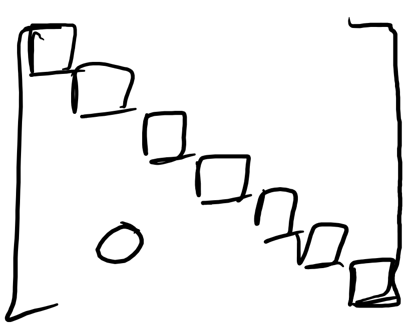
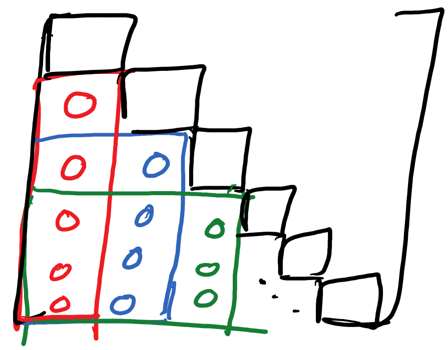
$[\varphi(e_1)]$ only needs

the first k vectors.

Suppose $\varphi : V \to V$ lin transf and

$e_1, \dots, e_n$ basis.

Then

$[\varphi]_{\underline{\underline{e}}} =$  $=$ 

$\Updownarrow$

dim $U_i = i$.

$U_1 := $ Span(e_1) is φ invariant

$U_2 := $ Span(e_1, e_2) is φ invariant

so

$U_3 := $ Span(e_1, e_2, e_3) is φ invariant

$\vdots$

$0 < U_1 < U_2 < \dots$

$< U_n = V$

$\Updownarrow$

maximal chain of subspaces (contains prior and dimension diff = 1)

max. chain of subspaces all φ-variant

Equivalent restatement of Thm:

If V is an n-dim space over $\mathbb{C}$,

then $\exists$ max. chain of subspaces that are

φ - invariant.

Obs. 1 - dim. invariant subspace always exists.

(Characteristic poly. has a root by Fund. Theorem of Algebra over $\mathbb{C}$, (assuming scalars $= \mathbb{C}$)

so eigenvalue exists $\Rightarrow$ eigenvector exists

$\Rightarrow$ span of eigenvector is 1-dim. invariant subspace.)

This does not necessarily hold in $\mathbb{R}$

(e.g. rotation - no eigenvector.)

DO If n even, $\exists\ \varphi : \mathbb{R}^n \to \mathbb{R}^n$ with no eigenvector.

$\boxed{\text{HW}}$ If n odd and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, φ has an eigenvector

Rest of proof for next time.

$\boxed{HW}$ (for Wed.) Prove that diagonalizable matrices are <u>dense</u> in $M_n(\mathbb{C})$;

<u>i.e.</u> $(\forall A \in M_n(\mathbb{C}))(\forall \epsilon > 0)(\exists B \in M_n(\mathbb{C}))$

(B is diagonalizable and $(\forall i, j)$

$(|a_{ij} - b_{ij}| < \epsilon))$

(Proof 5 lines.)

$f(t) = t^2 + 1$

$f(A) = A^2 + I$

$(\forall A \in M_n(\mathbb{F}))$

$(\exists f \in \mathbb{F}[x],\ f \neq 0)(f(A) = 0)$

$f(t) = a_0 + a_1 t + \dots + a_k t^k$

find $a_0, a_1, \dots, a_k$ s.t. not all 0.

want $a_0 I + a_1 A + a_2 A^2 + \dots + a_k A^k = 0$

$\downarrow$

nontrivial lin. comb. of powers of A ..

$(I, A^1, A^2, \dots, A^k)$

<u>Claim</u>: $(\exists k)(A^0, A^1, A^2, \dots, A^k$ are lin. dep.)

Claim: $k = n^2$ works.

$\boxed{\text{HW}}$ Prove this $\rightarrow$ 2 lines, using only what we have proved already

(n^2+1 matrices)