$\text{rank}(A) \le r \iff A$ is the sum of r rank 1 matrices.

($\Leftarrow$) If $A = B_1 + B_2 + \dots + B_r$ with each B_i rank 1, then

$$\text{rank } A \le \sum_{i=1}^{r} \text{rank}(B_i) = r.$$

($\Rightarrow$) ex.

$$2 = \text{rank}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{rank } 1}$$

If rank $< r$, you can pad the sum (divide matrices in 2, for ex.)

If $\text{rank}(A) = k \le r$, use elementary operations to get A in the form

matrix of elementary operations

$$\begin{array}{c} k \text{ lin indep rows} \end{array} \begin{bmatrix} \text{---}\; r_1 \;\text{---} \\ \text{---}\; r_2 \;\text{---} \\ \text{---}\; r_3 \;\text{---} \\ 0 \end{bmatrix} = A' = EA$$

$$A' = \begin{bmatrix} \text{---}\; r_1 \;\text{---} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \text{---}\; r_2 \;\text{---} \\ 0 \end{bmatrix} + \dots$$

s.t.?

$A = E^{-1}A'$ ← sum of k rank 1 matrices

Circulant matrix

$$C = C(a_0, a_1, \dots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & \cdots & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & a_{n-1} & a_0 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_0 \end{pmatrix}$$

What is $\det C$?

Hint: Express C as a polynomial of cyclic shift matrix

$$A = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & 0 & \\ & 1 & 0 & 0 & \\ & & 1 & 0 & \\ 0 & & & 1 & 0 \end{bmatrix}.$$

$C = a_0 I + a_{n-1} A + \dots + a_1 A^{n-1}$.

Find eigenvalues → (ω_k for A – roots of unity $\omega_k = e^{\frac{2\pi k}{n} i}$).

$\lambda_k = a_0 + a_{n-1}\omega_k + \dots + a_1 \omega_k^{n-1}$

$$\det C = \prod_{k=0}^{n-1} (a_0 + a_{n-1}\omega_k + \dots + a_1 \omega_k^{n-1})$$

Note: $\lambda_1, \dots, \lambda_n$ are eigenvalues of A ⇒ $f(\lambda_1), \dots, f(\lambda_n)$ are eigenvalues of $f(A)$.

D - invariant subspaces of $\mathbb{R}^{\le n}[t]$ are all

$\mathbb{R}^{\le t}[t]$ for $k = -\infty, 0, 1, \dots, n$

Derivative operator:

Note: $\mathbb{R}^{\le k}[t]$ is a D-invariant subspace

<u>Claim:</u> $\mathbb{R}^{\le k}[t]$ are the <u>only</u> D - invariant subspaces.

$W \le \mathbb{R}^{\le n}[t]$. $\exists$ highest degree of polynomials in W.

If there is a polynomial of degree d, by applying addition, scalar multiplication, and diff we get $\mathbb{R}^{\le d}[t]$.

If $0 \in W$, $\mathbb{R}^{\le -\infty}[t] = \{0\} \subset W$.

A polynomial p of degree $d \to$ diff $\to$ degree $d-1$ ...

by inductive hyp $\mathbb{R}^{\le d-1} \subset W$,

$$p = a_d t^d + a_{d-1} t^{d-1} + \dots + a_1 t + a_0$$
$$- a_{d-1} t^{d-1} - \dots - a_1 t - a_0$$

$a_d t^d$ } scaler mult

$c t^d$ } addition

$c t^d +$ (poly of degree $d-1$)

$\therefore \quad \mathbb{R}^{\le d}[t] \subset W$. Cannot increase degree,

so $W = \mathbb{R}^{\le d}[t]$ if d is max. $\square$

Any strictly upper triangular matrix is nilpotent

↳ upper triangular + diagonal entries 0

$(\exists k)$
$(N^k = 0)$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 = \underline{\underline{0}}.$$

Cayley - Hamilton. -

Since N is strictly upper triangular, all eigenvalues are 0 (on diagonal), so

$f_N(t) = t^n \quad (N \in M_n(\mathbb{F}))$.

$f_N(N) = N^n = \underline{\underline{0}}$ by Cayley - Hamilton, so

N is nilpotent $\square$

Alternate proof.

Let $U_i = \text{span}\{e_1, e_2, \ldots, e_i\}$ for $1 \leq i \leq n$. $(A$ is $n \times n)$ Let $U_0 = \{0\}$.

$U_0 \leq U_1 \leq U_2 \leq \ldots \leq U_n$.

$$A = \begin{bmatrix} 0 & & * \\ & 0 & \\ & & 0 & \\ 0 & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

what is AU_1?

$\downarrow$

what is Ae_1? 0.

Anything in AU_1 is a scalar multiple of Ae_1,

so $AU_1 = U_0$.

What is AU_2?

$$Ae_2 = \begin{bmatrix} 0 & & * \\ & 0 & \\ & & 0 \\ 0 & & & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = ce_1 \in U_1$$

$$AU_2 \subseteq U_1$$

Conjecture: $AU_i \subseteq U_{i-1}$

Inductively, we apply this n times:

$$A^n U_n \subseteq U_0 = \{0\}.$$

so $A^n U_n = \{0\}$ (0 needs to be in every subspace)

$\Rightarrow A^n \underline{v} = \underline{0}$ $\forall$ vectors v ...

so A^n must be the zero matrix and

A is nilpotent $\square$

N is nilpotent $\Leftrightarrow$ N is similar to a strictly upper triangular matrix.

($\Rightarrow$) N is similar to an upper triangular matrix A. Let $N\underline{v} = \lambda\underline{v}$. Then $\exists\, k$ s.t. $N^k = 0$ and $N^{k-1} \neq 0$.

Since λ is eigenvalue of N, λ^k is eigenvalue of N^k:

$$N^k\underline{v} = \lambda^k\underline{v}$$

$$0 = 0\underline{v} = \lambda^k\underline{v}$$

since $\underline{v} \neq \underline{0}$, $\lambda^k = 0$, so $\lambda = 0$.

Thus all eigenvalues of N are 0.

Diagonal of A has the eigenvalues, so diagonal entries are all 0.

$(\Leftarrow)$

$$A = \begin{bmatrix} 0 & & * \\ & 0 & \\ & & \ddots \\ 0 & & & 0 \end{bmatrix}$$

$f_A(t) = t^n$

$N \sim A$ so $f_N(t) = t^n$ as well

Then N is nilpotent by previous exercise □

(you can show $f_N(t) = t^n \Rightarrow N$ nilpotent

using Cayley - Hamilton).

$N^k = S^{-1}A\cancel{SS^{-1}}A\cancel{S} \cdots \cancel{S^{-1}}AS$

$A^k = 0$

$= S^{-1}N\cancel{SS^{-1}}N\cancel{S \cdots S^{-1}}NS = S^{-1}A^kS$

$N^k = 0.$

$= S^{-1}N^kS = 0 \qquad = 0$

$SS^{-1}N^kSS^{-1} = S0S^{-1}$ $\therefore N$ is nilpotent

If $BC = 0$ and B invertible, $C = 0$.

$B^{-1}BC = B^{-1}0 = 0$

$IC = 0 \Rightarrow C = 0$

Class Tues / Thurs $\Rightarrow$ Ryerson 251. (easier to see)

<u>Lecture</u>

If $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ ($\lambda_i \in \mathbb{R}$ $\forall i \in [n]$)

and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1}$ ($\mu_j \in \mathbb{R}$ $\forall j \in [n-1]$),

we say that the μ_i <u>interlace</u> the λ_j

if

$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n$.

e.g. λ: 15 10 9 9 9 3

μ: 14 9 9 9 4

DO If $f \in \mathbb{R}[x]$ s.t. all roots of f are real ($f(x) = a_n \prod_{i=1}^{n} (x - \lambda_i)$ where $\lambda_i \in \mathbb{R}$ $\forall i \in [n]$), then all roots of f' are real and they interlace the roots of f.

$\boxed{\text{Do*}}$ If $\mathbb{F}$ is a finite field of order q then $(\forall a \in \mathbb{F})(a^q = a)$.

Special case of this: <u>Fermat's Little Theorem</u>:

$(\forall a \in \mathbb{Z})(a^p \equiv a \mod p)$ (p prime)

<u>i.e.</u> in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ we have $a^p = a$ $\forall a \in \mathbb{F}_p$.

DO Prove <u>Fermat's Little Theorem</u> by induction on <u>a</u> $(a \geq 0)$.

(Easy based on previously assigned exercise)

$A \in \mathbb{Z}^{k \times \ell}$ <u>integral matrix</u>

Rank over $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ all same -- why?

show $rk_{\mathbb{Q}}(A) = rk_{\mathbb{R}}(A)$. (HW prob.)

$\geq$ (clear b/c if $v_1, \dots, v_k$ lin. indep. over $\mathbb{R}$ $\Rightarrow$ lin. indep. <u>over $\mathbb{Q}$</u>.) $v_i \in \mathbb{Z}^k$

$\leq$?

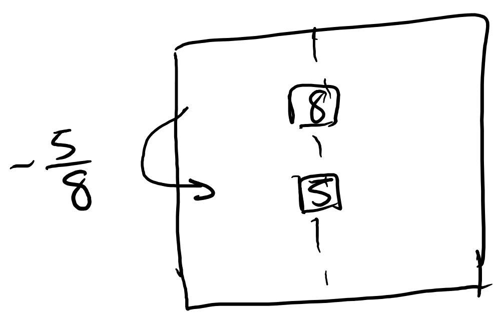
$\mathrm{rk}(A) = \max\{s \mid \exists\ s\times s$ <u>nonsingular</u> submatrix$\}$

Does nonsingularity change with the field?

No — the determinant does not change with the field (doing the same computations either way).

So exactly the same square submatrices are non-singular over $\mathbb{Q}$ as over $\mathbb{R}$.

Thus $\mathrm{rank}_{\mathbb{Q}}(A) = \mathrm{rank}_{\mathbb{R}}(A)$



elimination — takes place over $\mathbb{Q}$

end up w/ at most 1 nonzero entry in every row/col. — # of nonzero entries is rank

$A \in \mathbb{Z}^{k \times l}$ integral matrix

Rank over $\underbrace{\mathbb{Q}, \mathbb{R}, \mathbb{C}}_{\text{characteristic zero}}$ all the same

$rk_0(A)$

$rk_p(A)$: rank over $\mathbb{F}_p$

Find a matrix s.t. $rk_0(A) \neq rk_p(A)$.

$$\begin{bmatrix} 0 & p & 0 \\ 0 & p & p \\ 0 & 0 & 0 \end{bmatrix}.$$

$rk_0(A) \neq 0$, but $rk_p(A) = 0$.

$\boxed{HW}$ Find 3×3 $\underbrace{(0,1)}_{\text{every entry 0 or 1}}$-matrix A s.t.

$rk_2(A) < rk_0(A)$.

$\boxed{HW}$ $\forall A \in \mathbb{Z}^{k \times l}$, $rk_p(A) \leq rk_0(A)$.

Cyclic shift matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad e_1 \mapsto e_2 \mapsto e_3 \mapsto \cdots \mapsto e_n \mapsto e_1$$

eigenbasis $\mathbb{C}$

n^{th} roots of unity $\rightarrow$

$$\begin{pmatrix} 1 & & 1 & \\ 1 & & \omega_k & \\ 1 & \cdots & \omega_k^2 & \cdots \\ 1 & & \vdots & \\ 1 & & \omega_k^{n-1} & \end{pmatrix}$$

discrete Fourier transform matrix.

$\omega_k^{n-1} \rightsquigarrow \lambda = \omega$

$\omega_1 = e^{\frac{2\pi i}{n}}$

$= V(1, \omega_1, \omega_1^2, \ldots, \omega_1^{n-1})$

$\uparrow$ Vandermonde

eigenvalues : n^{th} roots of unity.

\# of invariant subspaces of φ (assuming n distinct eigenvalues) $= 2^n$ — $\exists\, 2^n$ invariant subspaces — these are the only ones.

$\lambda_1, \ldots, \lambda_n$ $\quad \forall I \subseteq [n]$,

$\varphi e_i = \lambda_i e_i$ $\quad U_I = \text{span}(e_i \mid i \in I)$ is a φ-invariant subspace

There exist 2^n such I (power set of n.)

<u>Claim:</u> If $U \leq V$ with $\varphi : V \to V$ and

$\varphi(U) \subseteq U$ (U is φ-invariant), then

$(\exists I \subseteq [n])(U = U_I)$.

$\bar{\varphi} := \varphi|_U$ Since over $\mathbb{C}$, $\bar{\varphi}$ has an

eigenvector b/c char. poly.

$\bar{\varphi} : U \to U$ must have a root by

Fund Thm of Algebra.

<u>Claim.</u> $\bar{\varphi}$ has an eigenbasis.

Suppose $\varphi : V \to V$ and U is φ-invar. subspace.

$\bar{\varphi} = \varphi|_U$

$\boxed{\text{HW}}$ (for Mon.) $f_{\bar{\varphi}} \mid f_{\varphi}$.

char. poly.

Since $f_{\bar{\varphi}} \mid f_{\varphi}$, all roots of $f_{\bar{\varphi}}$ are distinct,

so $\bar{\varphi}$ has an <u>eigenbasis</u>.

These eigenvectors are eigenvectors of φ as well.

Suppose f is an eigenvector other than scalar multiples of e_i.
$\Uparrow$
eigenvectors of φ.

$\varphi f = \lambda f \Rightarrow (\exists\, i)(\lambda = \lambda_i)$ (cannot add more eigenvalues.)

<u>Claim</u> $f \in \operatorname{Span}(e_i)$.

Suppose otherwise:

$$\left.\begin{array}{ll} f & \lambda_i \\ e_i & \lambda_i \end{array}\right\} \text{not scalar multiples.}$$

It follows that geom. mult $(\lambda_i) \geq 2$, but alg. mult. $(\lambda_i) = 1$ (all eigenvalues distinct) and geom mult $\leq$ alg mult., so contradiction.

Then f must be scalar multiples of e_i.

Thus eigenvectors of eigenbasis of $\bar{\varphi}$ are scalar multiples of some e_i's $\Rightarrow$ Span of these e_i's — $U = U_I$ for some $I \subseteq [n]$. $\square$.

$\boxed{HW}$ (for Mon.)

$A \in M_p(\mathbb{F}_p)$ $p \times p$

cyclic shift

<u>Prove:</u> # of invariant subspaces is $p+1$ and they form max. chain

Char. poly. of A.

$f_A(t) = t^p - 1$ (in $\mathbb{C}$)

$= (t-1)^p$ (over $\mathbb{F}_p$ from previous exercise)

Thus, eigenvalue is 1 w/ alg. mult p.

$$J_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n \times n}$$ all-ones.

$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n \times n}$$ identity matrix.

$$J_n - I_n = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$$

DO $rk(J_n - I_n) = n$

DO Find eigenvalues, eigenbasis. <u>$rk\, J = 1$</u>

$\varphi : V \to V$

$rk(\varphi) = 1 = \dim \operatorname{Im}(\varphi) = 1$

$\dim(\operatorname{Ker} \varphi) = n - 1$ (by rank-nullity thm)

$v \in \operatorname{Ker} \varphi \Rightarrow \varphi(v) = 0$, i.e. it is eigenvector w/ eigenvalue 0.

$\therefore$ $n-1$ lin. indep. eigenvectors to eigenvalue 0

($\dim(\operatorname{Ker} \varphi) = n - 1$), so

geom mult$_\varphi(0) = n - 1. \leq$ alg mult$_\varphi(0)$

but $J \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = n \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,

so eigenvalues of $J \to (\underbrace{0, \dots, 0}_{n-1}, n)$.

$n-1$ lin. indep. eigenvectors + lin indep to distinct value

$\Rightarrow$ eigenbasis.

(or geom. mult = alg mult)

$A\underline{e} = \lambda \underline{e}$

$\Downarrow$

$(A - I)\underline{e} = (\lambda - 1)\underline{e}$

So $J - I$ has an eigenbasis as well,

and eigenvalues of $J - I$:

$(-1, \ldots, -1, n-1)$.

$\boxed{\text{HW}}$ $rk_2(J_n - I_n)$ (answer depends on n).
(sometimes same as rationals, sometimes not).

$rk_{\mathbb{F}}(A+B) \leq rk_{\mathbb{F}}(A) + rk_{\mathbb{F}}(B)$

If $B = -A$, $rk_{\mathbb{F}}(A - A) = 0$ but $rk_{\mathbb{F}}(A) \neq 0$.

$n-1$

$? \leq rk_2(J_n - I_n) \leq n$

$\boxed{\text{HW}}$ $|rk(A) - rk(B)| \leq rk(A+B)$

Clubtown

n residents

infinite – distinct clubs

can have same membership.

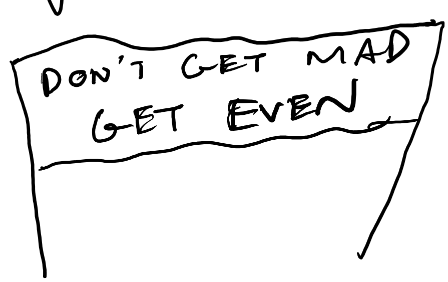
Rule 0. no two clubs can have identical membership.

Now $\leq 2^n$ clubs.

Rule 1. Every club must have an even # of members.

$\leq 2^{n-1}$ clubs.

Rule 2. Every pair of clubs shares an even # of members. (Eventown)



$\leq 2^{\lfloor n/2 \rfloor}$ clubs

"married couples"

solution – pair up people + make them join the same clubs.

$\boxed{\text{CH}}$ Is this best possible for Eventown?

Oddtown.

① Every club.. odd # of people

② Every pair of clubs share even # of people

smallest # of people in clubs: 1

①, ②, ..., ⓝ n clubs – "singles clubs"

# of Oddtown clubs $\leq n$.

$\boxed{\text{HW}}$ (for Mon.)

$\boxed{\text{CH}}$ $\exists\ 2^{\Omega(n^2)}$ oddtown club systems

at least cn^2 (lower bound).