Monday, July 17, 2017 9:35 AM

$\boxed{\text{HW}}$ If G is a k-regular grap of girth ≥ 5, then $n \geq k^2+1$.

(girth = length of shortest cycle)

$\downarrow$ no cycles of length ≤ 5.

girth(tree) = $\min \emptyset = \infty$

<u>Note:</u> $A \subseteq B \subseteq \mathbb{R}$

$\min A \geq \min B$

$\forall A$ $\min \emptyset \geq \min A$

$\geq \min \{10^6, 10^6+1\}$

so $\min \emptyset = \infty$.

DO $f \in \mathbb{Z}[t]$ $f(t) = a_0 + a_1 t + \cdots + a_n t^n$

$a_0 \neq 0$ and $a_n \neq 0$.

suppose $f(\alpha) = 0$ and $\alpha = \frac{r}{s}$ where $\gcd(r,s) = 1$.

Then $s \mid a_n$ and $r \mid a_0$.

$\mathbb{R}$ real quadratic forms over $\mathbb{R}^n$.

$\underline{x} \in \mathbb{R}^n$ $A \in M_n(\mathbb{R})$, $\underbrace{A = A^T}_{\text{symmetric}}$

$$q(x) = \underline{x}^T A \underline{x} = \sum_{i,j} a_{ij} x_i x_j .$$

If $(\forall \underline{x})(q(\underline{x}) \geq 0)$ then we call q "<u>positive semidefinite</u>".

In this case we also call A "<u>positive semidefinite</u>".

If $(\forall \underline{x} \neq 0)(q(\underline{x}) > 0)$ then we call q, A "<u>positive definite</u>."

If $(\forall \underline{x})(q(\underline{x}) \leq 0)$ then we call q, A "<u>negative semidefinite</u>."

If $(\forall \underline{x} \neq 0)(q(\underline{x}) < 0)$ then we call q, A "<u>negative definite</u>."

If q is neither positive semidefinite nor negative semidefinite, we call q, A "<u>indefinite</u>".

i.e. q is indefinite if

$$(\exists \underline{x}, \underline{y} \in \mathbb{R}^n)(q(\underline{x}) > 0,\ q(\underline{y}) < 0).$$

$\|\underline{x}\| = \sum x_i^2$ (Euclidean norm) - positive definite.

$x_1^2 + x_2^2 + x_3^2 - x_4^2$ indefinite (special relativity metric)

DO $A \in M_n(\mathbb{R})$, symmetric.

A is positive definite $\Leftrightarrow$ all eigenvalues are positive.

Proof Spectral Thm. $\Rightarrow$ A has orthonormal eigenbasis.

$\hookrightarrow e_1, \ldots, e_n$ s.t. $Ae_i = \lambda_i e_i$ - eigenbasis

$$e_i \cdot e_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$ orthonormal.

$\underline{x} = \sum \alpha_i e_i$

$\underline{x}^T A \underline{x} = \sum \lambda_i \alpha_i^2$

If all positive, will definitely be $+$.

If not all positive, can choose vector s.t. ≤ 0. $\square$

pos. semidefinite $\iff (\forall i)(\lambda_i \geq 0)$

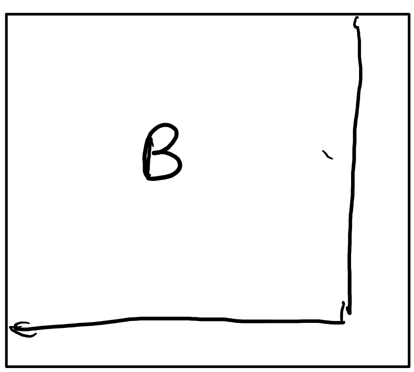
neg. def $\iff (\forall i)(\lambda_i < 0)$

neg. semidefinite $\iff (\forall i)(\lambda_i \leq 0)$

indefinite : $(\exists i, j)(\lambda_i > 0$ and $\lambda_j < 0)$

Observation : If A pos. def. then $\det A$ is positive.

(det is product of eigenvalues - positive)



A

$B = {}_{\hat{n}}A_{\hat{n}}$

$\Uparrow$

$n-1 \times n-1$ symmetric matrix.

Observation If A is positive definite, then B is positive definite.

Claim : If $y \in \mathbb{R}^{n-1}$ and $y \neq 0$ then $y^T B y > 0$.

Let $\underline{x} := \begin{bmatrix} \underline{y} \\ 0 \end{bmatrix}$ $\leftarrow$ n-1 coordinates

$$\underline{y}^T B \underline{y} = \underline{x}^T A \underline{x} > 0$$

(0 does not contribute to the sum.) $\square$

__Cor.__ All "corner matrices" (cut off last row/last column) are positive definite

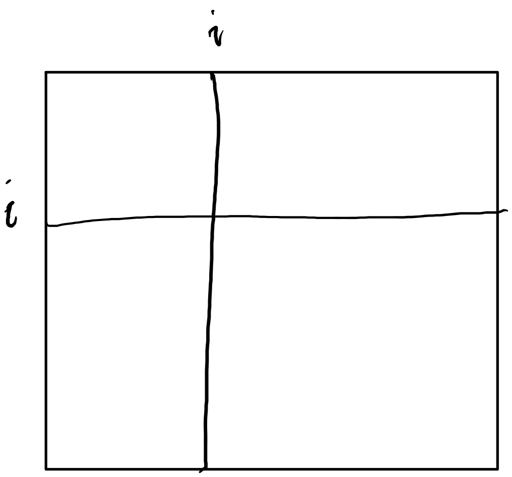
__Cor.__ All corner matrices have positive determinant

__Thm__ A pos. def. $\iff$ all corner matrices have positive determinant

DO

DO If A pos. def, then all diagonal entries are positive: $a_{ii} > 0$.

(Hint: the eigenvalues interlace.)



$_{\hat{i}}A_{\hat{i}}$ pos. def

$2^n - 1$ possible symmetric matrices.
(but by thm, n conditions suffice.)

$-I_{2\times 2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

DO Find A s.t. all corner det ≥ 0 yet A is <u>indefinite</u>.

$-\sum x_i^2$

Question: How can we check (without using eigenvalues) whether A is positive semidefinite?

<u>Spectral Graph Theory</u>

$G \rightsquigarrow$ adjacency matrix $A_G = (a_{ij})$

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{ (adjacent)} \\ 0 & \text{o/w} \end{cases}$$

$a_{ii} = 0 \ \forall i$

$\therefore \operatorname{Tr} A = 0$

$\therefore \sum_i \lambda_i = 0.$

$\lambda_1 \geq \dots \geq \lambda_n$

avg. deg. $\leq \lambda_1 \leq$ max deg.

$\therefore$ If G is k-regular,

$\lambda_1 = k$.

i — 0 1 1 0 1 0 —

sum of i^{th} row = deg of vertex i

Eigenvector to λ_1: all-ones $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \deg(1) \\ \vdots \\ \deg(n) \end{pmatrix}$$

Complete graph

$$A_{K_n} = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & \ddots \; 0 \end{pmatrix} = \underbrace{J_n}_{\text{all ones}} - \underbrace{I_n}_{\text{identity.}}$$

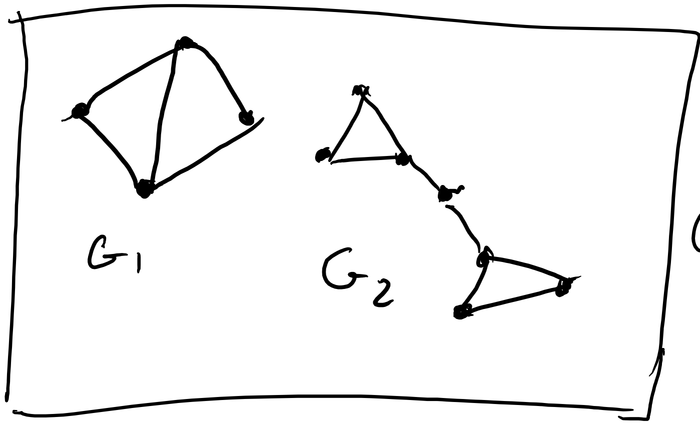
geom mult$_J(0) = n-1$

$\therefore$ n^{th} eigenvalue of J $= \underline{n}$ (since $Tr(J) = n$)

J: $n, 0, \ldots, 0$

subtracting I from a matrix reduces the eigenvalues by 1, so

$J - I$: $n-1, -1, \ldots, -1$.



$$\begin{array}{c} G_1 \\ G_2 \end{array} \left[\begin{array}{c|c} A_{G_1} & 0 \\ \hline 0 & A_{G_2} \end{array}\right]$$

disjoint union $\to G$

$f_G = f_{G_1} \cdot f_{G_2}$ so eigenvalues of G are combined eigenvalues of G_1, G_2

char poly of adj. matrix of G

Suppose G_1, G_2 are k-regular. Then k has multiplicity ≥ 2.

$$\begin{array}{c} G_1 \\ G_2 \end{array} \left[\begin{array}{c|c} A_{G_1} & 0 \\ \hline 0 & A_{G_2} \end{array}\right] \qquad \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \end{bmatrix}}_{\text{lin indep. to eigenvalue } k}$$

DO Assume G is k-regular. Then

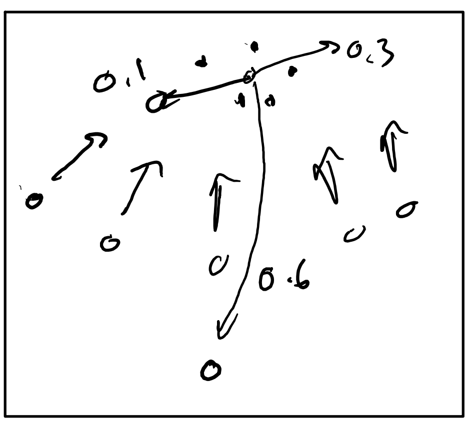
$\lambda_2 = k \iff G$ is disconnected

In fact, multiplicity of k = # connected components.

For k-regular graph, "as the eigenvalue gap $\lambda_1 - \lambda_2 = k - \lambda_2$ grows, the graph gets more interconnected." → "algebraic connectivity of the graph" (discovered by Miroslav Fiedler ~1970s) → $\underline{\underline{K_{n-1}}}$

Mixing rate of random walks, Markov chains.

Scottland Yard - British FBI (board game)



London

Every 5^{th} move - villain surfaces - probability 1.

Then diffuses ... converges toward uniform/ until reappearance

stationary distributions.

Laplacian of graph

$L_G = D_G - A_G$ where $D_G = \begin{pmatrix} \deg 1 & & 0 \\ & \deg 2 & \ddots \\ 0 & & \deg(n) \end{pmatrix}$

$\uparrow$ adjacency matrix

Thm L_G is positive semidefinite.

Claim. $\det(L_G) = 0$. $L_G = \begin{pmatrix} \deg 1 & 0 & -1 & -1 & 0 \\ 0 & \deg 2 & -1 & 0 & -1 \\ & \ddots & & \ddots & \\ & & & & \deg n \end{pmatrix}$

(0 is an eigenvalue w/ eigenvector all-ones.) sum of each row is 0.

Obs. A positive definite matrix is nonsingular, so L_G is not positive definite.

Thm If L_G has eigenvalues $\kappa_1 \geq \cdots \geq \kappa_n$, then $\kappa_n = 0$.

\kappa

Proof Singular and positive semidefinite. $\square$

When is multiplicity of $0 > 1$?

Empty graph, isolated vertex... $\boxed{\text{disconnected graph}}$.

DO Multiplicity of 0 = # of connected components.

In particular, $\underline{\kappa_{n-1} > 0} \iff G$ connected

If G is k-regular, then

$L_G = kI - A_G$ If eigenvalues of A_G are $\lambda_1 \geq \dots \geq \lambda_n$,

$$0 = \underbrace{k - \lambda_1}_{\kappa_n} \leq \underbrace{k - \lambda_2}_{\kappa_{n-1}} \leq \dots \leq \underbrace{k - \lambda_n}_{\kappa_1}.$$

Show: L_G is positive semidefinite.

DO $\underline{x}^T L_G \underline{x} = \sum_{i \sim j} (x_i - x_j)^2$ $\underline{x} \in \mathbb{R}^n$ $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

≥ 0

Also solves multiplicity of 0 = # connected components.

Euclidean space:

V over $\mathbb{R}$ endowed with a positive definite

inner product:

bilinear form $\underline{x}, \underline{y} \in V \longmapsto \langle \underline{x}, \underline{y} \rangle \in \mathbb{R}$

\langle $\underline{x}, \underline{y}$ \rangle

$\downarrow$

$\langle \underline{x} + \underline{y}, \underline{z} \rangle = \langle \underline{x}, \underline{z} \rangle + \langle \underline{y}, \underline{z} \rangle$

etc.

symmetric: $\langle \underline{x}, \underline{y} \rangle = \langle \underline{y}, \underline{x} \rangle$

positive definite: $(\forall \underline{x} \in V, \underline{x} \neq \underline{0})(\langle \underline{x}, \underline{x} \rangle > 0)$.

Def $\underline{x} \perp \underline{y}$ if $\langle \underline{x}, \underline{y} \rangle = 0$.

$\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle}$

DO Cauchy-Schwarz: $|\langle \underline{x}, \underline{y} \rangle| \leq \|\underline{x}\| \cdot \|\underline{y}\|$

DO Triangle Inequality: $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$.

Standard example : $V = \mathbb{R}^n$

$$\langle \underline{x}, \underline{y} \rangle = \underline{x} \cdot \underline{y}$$

General form of bilinear forms over $\mathbb{F}^n$:

$$f(\underline{x}, \underline{y}) = \underline{x}^T A \underline{y}. \qquad A \in M_n(\mathbb{F})$$

$$\underbrace{f(\underline{x}, \underline{y}) = f(\underline{y}, \underline{x})}_{f \text{ symmetric}} \iff \underbrace{A = A^T}_{A \text{ symmetric}}$$

$f(\underline{x}, \underline{x})$ is positive definite $\iff$ A pos. def. (over $\mathbb{R}$)

General form of a positive definite inner product over $\mathbb{R}^n$ is

$\langle \underline{x}, \underline{y} \rangle = \underline{x}^T A \underline{y}$ where A is pos. def and symmetric

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$f, g \in \mathbb{R}[x]$ (real polynomials)

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \cdot g(t) \cdot e^{-t^2} \, dt$$

DO $\langle f, g \rangle$ always converges.

This is positive def

More generally,

$$\langle f, g \rangle = \int f \cdot g \cdot w(t)\, dt$$

w weight function ≥ 0

sometimes > 0.

(also integral must converge)

Matrices

$A, B \in M_n(\mathbb{R})$ DO pos. def

ex: $\langle A, B \rangle = Tr(A^T B)$

Gram - Schmidt orthogonalization

input: $v_1, v_2, v_3, \ldots$ (sequence of vectors)

output: $b_1, b_2, b_3, \ldots$

s.t. ① For $i \neq j$, $b_i \perp b_j$.

② $b_i - v_i \in Span(v_1, \ldots v_{i-1})$.

$v_1 \rightarrow \boxed{} \rightarrow b_1$

$v_2, v_1 \rightarrow \boxed{} \rightarrow b_1, b_2$

Thm ① and ② uniquely determine the output

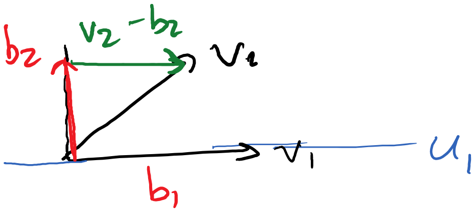
Proof Let $U_i = \text{span}(v_1, \dots, v_i)$

$U_0 = \text{Span}(\emptyset) = \{\underline{0}\}$

Claim. $U_i = \text{Span}(b_1, \dots, b_i)$

Suppose already $U_{i-1} = \text{Span}(b_1, \dots, b_{i-1})$.

DO (from ②)



v_3 : parallel plane to paper at tip.

choose ⊥ to paper, tip on plane for b_3.

$b_1 - v_1 \in U_0 \Rightarrow b_1 = v_1$

Now we need to find b_i in the form

$$b_i = v_i + \sum_{j<i} \alpha_j b_j .$$

need to find α_j, and need $b_i \perp b_k$ for $k < i$.

$$\langle b_i, b_k \rangle = \langle v_i, b_k \rangle + \sum_{j=1}^{i-1} \alpha_j \underbrace{\langle b_j, b_k \rangle}_{0 \text{ unless } j=k}$$

we wish $= 0$.

$$= \langle v_i, b_k \rangle + \alpha_k \|b_k\|^2$$

$$\alpha_k = -\frac{\langle v_i, b_k \rangle}{\|b_k\|^2}$$

LOGIC

$\Downarrow$ uniqueness

$\Uparrow$ existence (reversible)

$$\alpha_k = - \frac{\langle v_i, b_k \rangle}{\|b_k\|^2}$$

what if $\|b_k\|^2 = 0$?

Then $b_k = 0$.

but then α_k does not matter.

(choose $\alpha_k = 75.3$ or $\alpha_k = 0$.)

DO If $v_1, v_2, \ldots$ are pairwise orthogonal, nonzero, then lin. indep

when is $b_k = 0$? DO

$\Leftrightarrow v_k \in \mathrm{Span}(v_1, \ldots, v_{k-1})$

In particular, If the v_i are linearly independent then none of the b_i are 0.

$\Rightarrow b_i$ are linearly independent

In particular if $v_1, \dots, v_n$ is a basis then $b_1, \dots, b_n$ is also a basis.
$\hookrightarrow$ orthogonal

$$\underline{v} \neq \underline{0} \Rightarrow \underline{v}' = \frac{\underline{v}}{\|\underline{v}\|} \Rightarrow \|\underline{v}'\| = 1.$$

<u>Cor.</u> If dim V is finite [or countable], then V has an orthonormal basis.

(Take basis $\to$ apply Gram-Schmidt $\to$ normalize outputs)

Basis of $\mathbb{R}[t]$: $1, t, t^2, \dots$

if we have a weight function $w(t)$ s.t.

$$\int_{-\infty}^{\infty} t^{2n} w(t)\, dt < 0 \quad \text{for every } n,$$

we can orthogonalize $1, t, t^2, \dots$ to get a seq. of orthogonal polynomials $f_0, f_1, f_2, \dots$ where $\deg f_i = i$.

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f \cdot g \cdot w \, dt \qquad (\forall i)(\operatorname{span}(f_0, \dots, f_i) = \mathbb{R}^{\leq i}[t])$$

Fix weight function w.

$f_0, f_1, \ldots$ - sequence of orthogonal polynomials.

Thm. All roots of the f_i are real and
the roots are interlaced. DO
(roots of f_{i-1} interlace roots of f_i).

$\cos(nt) = T_n(\cos t)$ (polynomial of deg n)

$\cos(2t) = 2\cos^2 t - 1$ $T_1(x) = x$

$T_2(x) = 2x^2 - 1$

DO Evaluate T_n for $n = 3, 4, 5$.

T_n are called the Chebyshev polynomials of the first kind.

$$\frac{\sin(n+1)t}{\sin t} = U_n(\cos t)$$

Chebyshev polynomials of second kind.

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$n=0 \qquad U_0(x) = 1$

$n=1 \qquad U_1(x) = 2x \qquad \sin 2t = 2\sin t \cos t$

DO Evaluate for small n.

These polynomials are orthogonal wrt.

$$\langle f, g \rangle = \int_{-1}^{1} f \cdot g \begin{cases} \sqrt{1-t^2} \\ \dfrac{1}{\sqrt{1-t^2}} \end{cases} dt.$$

Hermite polynomials

weight function : $e^{-t^2} \qquad e^{-t^2/2}$

matching polynomials – physicists.