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DO

$$f(t) = \frac{at^2 + bt + c}{dt^2 + e} \qquad d, e > 0$$

Suppose $(\forall t \in \mathbb{R})(f(0) \geq f(t))$.

Then $\underline{b = 0}$.

DO $A, B \in \mathbb{F}^{k \times \ell}$

$A = B \iff (\forall x \in \mathbb{F}^k)(\forall y \in \mathbb{F}^\ell)(x^T A y = x^T B y)$

$U \leq \mathbb{F}^n$

$\dim U + \dim U^\perp = n$

$\times$ $U \oplus U^\perp = V$

$U + U^\perp = V \iff U \cap U^\perp = \{0\}$

false if isotropic vectors.

DO If $\nexists$ isotropic vectors, then $U \oplus U^\perp = V$ (over all $\mathbb{F}$).

V Euclidean space (over $\mathbb{R}$)

$\dim V = n$

$x \perp y$ if $\langle \underline{x}, y \rangle = 0$

$S \subset V$ $\quad S^{\perp} = \{ \underline{v} \mid \underline{v} \perp S \}$

$\underline{v} \perp S$ if $(\forall \underline{s} \in S)(\underline{v} \perp \underline{s})$.

$\langle \underline{v}, \underline{v} \rangle = 0 \Rightarrow \underline{v} = 0$.

(b/c inner product is positive definite)

DO $\quad U \leq V \Rightarrow U \oplus U^{\perp} = V$

in particular, $\dim U + \dim U^{\perp} = n$.

DO $\quad$ This is false in infinite dimensions.

<u>Pythagorean Theorem</u>

If $\underline{a} \perp \underline{b}$ then $\|a+b\|^2 = \|a\|^2 + \|b\|^2$.

Pf. $\|a+b\|^2 = \langle a+b, a+b \rangle = \langle a,a \rangle + \langle b,b \rangle + \underbrace{\langle a,b \rangle + \langle b,a \rangle}_{0}$

$= \langle a,a \rangle + \langle b,b \rangle = \|a\|^2 + \|b\|^2$.

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DO $v_1, \ldots, v_k$ where $v_i \perp v_j$ $(i \neq j)$

$$\Rightarrow \|\sum v_i\|^2 = \sum \|v_i\|^2.$$

$\varphi : V \to W$ $\quad V, W$ Euclidean, finite dimensional.

Def $\psi : W \to V$ is a **transpose** of φ if:

\psi

$$(\forall x \in V)(\forall y \in W)(\langle \varphi x, y \rangle = \langle x, \psi y \rangle)$$

Thm. $\forall \varphi$ $\exists!$ transpose.

Proof Choose an ONB in V, W.

Condition above can be rewritten as

$$[\varphi x]^T [y] = [x]^T [\psi y].$$

wrt ONB, $\langle u, v \rangle = [u]^T [v]$

$[\varphi x] = [\varphi][x]$

$(AB)^T = B^T A^T$

$$([\varphi][x])^T [y] = [x]^T [\psi][y]$$

$$[x]^T [\varphi]^T [y] = [x]^T [\psi][y] \quad \forall x, y$$

Then $[\varphi]^T = [\psi]$ by previous DO exercise

LOGIC: $\Downarrow$ uniqueness $\quad \Uparrow$ existence.

Lemma If $\varphi : V \to V$ is a linear transformation, then $U \leq V$ is φ-invariant $\Rightarrow$ $U^\perp$ is ψ-invariant

$\uparrow$

φ^T

Proof

Assumption: $(\forall u \in U)(\varphi(u) \in U)$

Conclusion: $\underbrace{(\forall v \in U^\perp)}\underbrace{(\varphi^T(v) \in U^\perp)}$

$(\forall v)(\text{if } v \perp U \text{ then } (\forall w \in U)(\underbrace{\langle w, \varphi^T(v)\rangle} = 0))$

$\langle \varphi(w), v \rangle = 0$

$v \perp \varphi(w)$

D. C.

If $v \perp U$ then $v \perp \varphi(U)$.

But $\varphi(U) \subseteq U$ b/c U is φ-invariant $\square$

In finite-dimensional Euclidean space, $U^\perp$ is the "orthogonal complement" of U.

Def. $\varphi : V \to V$ is symmetric if

$\varphi = \varphi^T$; i.e. $(\forall x, y)(\langle \varphi x, y \rangle = \langle x, \varphi y \rangle)$.

DO φ is symmetric $\iff [\varphi]_{ONB}$ is symmetric.

(regardless of the choice of the ONB - if true for one ONB, true for all)

Proof $[\varphi^T] = [\varphi]^T$

$[\varphi] = [\varphi^T]$

$\iff [\varphi] = [\varphi]^T$ $\square$.

Cor. (Spectral Thm)

If $\varphi : V \to V$ symmetric lin. transformation then φ has ON eigenbasis.

V: finite-dim Euclidean space ($\mathbb{R}$)

Lemma: If $\varphi : V \to V$ symmetric and $\dim V \geq 1$, then $\exists$ eigenvector.

Proof. (of spectral Thm modulo Lemma)

By induction on $n = \dim V$:

base case: $n = 0 \Rightarrow \emptyset$ = ON eigenbasis

Assume $n \geq 1$ and thm true for $\dim \leq n-1$.

By lemma, $\exists$ eigenvector.

Divide by norm $\rightarrow$ $\underline{e_1}$ where $\|e_1\| = 1$ and

$\varphi e_1 = \lambda_1 e_1$.

$\mathcal{U} := \mathrm{span}(e_1) \rightarrow$ 1-dim φ-invariant subspace

If $\mathcal{U}$ is φ-invariant, then $\mathcal{U}^\perp$ is φ^T-invariant;

i.e. φ-invariant (b/c symmetric.)

$V = \mathcal{U} \oplus \mathcal{U}^\perp$

$\dim \mathcal{U}^\perp = n-1$ (by previous DO)

Let $\bar{\varphi} := \varphi|_{\mathcal{U}^\perp}$.

we wish to apply inductive hyp. to $\bar{\varphi}$.

Note $\bar{\varphi} : \mathcal{U}^\perp \rightarrow \mathcal{U}^\perp$, so $\bar{\varphi}$ is a linear transformation on $\mathcal{U}^\perp$.

restriction $\uparrow$ $\quad$ $\uparrow$ $\mathcal{U}^\perp$ is φ-invariant

<u>Claim</u>. $\bar{\varphi}$ is symmetric, i.e. $(\forall x, y \in \mathcal{U}^\perp)$
$(\langle x, \bar{\varphi} y \rangle = \langle \bar{\varphi} x, y \rangle)$

$\bar{\varphi}(y) = \varphi(y)$ and $\bar{\varphi}(x) = \varphi(x)$ and we know $(\forall x, y \in \mathcal{U}^\perp)$ $(\langle x, \varphi y \rangle = \langle \varphi x, y \rangle)$

by def. of restriction, and b/c φ symmetric and $\mathcal{U}^\perp \leq V$,

so $\bar{\varphi}$ symmetric as well

DO If V is Euclidean and $U \leq V$, then U is Euclidean wrt. same inner product restricted to U.

conditions satisfied $\Rightarrow$ apply ind hyp

By ind. hyp, $\exists$ ON eigenbasis of $\bar{\varphi}$:

$e_2, \ldots, e_n \in U^{\perp}$ s.t. $\bar{\varphi} e_i = \lambda_i e_i$

$$\left(\begin{array}{l} (\forall i \geq 2)(\|e_i\| = 1) \\ e_i \perp e_j \quad \text{if } i > j \geq 2. \end{array} \right.$$

$\therefore e_1, e_2, \ldots, e_n$ ON eigenbasis of φ

$i \geq 2$

$\varphi e_i = \bar{\varphi} e_i = \lambda_i e_i$.

$e_1 \perp e_i \quad i \geq 2 \quad$ b/c $e_i \in \operatorname{Span}(e_1)^{\perp} \; \forall i \geq 2$. $\square$

Now prove <u>Lemma</u> If $\varphi: V \to V$ symm lin transformation w/ dim $V \geq 1$, then $\exists$ eigenvector of φ.

<u>Proof 1.</u>

<u>Def</u> Rayleigh quotient: $\boxed{R_\varphi(\underline{x}) = \dfrac{\langle x, \varphi x \rangle}{\langle x, x \rangle}}$ $\to \|x\|^2$

$\underline{x} \in V$
$\underline{x} \neq 0$

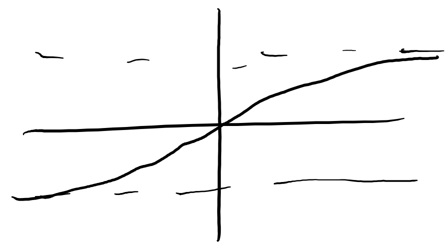
(note: $R_A(x) = \dfrac{x^T A x}{x^T x}$)

<u>Sublemma.</u> $R_\varphi(\underline{x})$ <u>attains</u> its maximum;

<u>i.e.</u> $(\exists\, \underline{x}_0 \neq 0)(\forall \underline{x} \neq 0)(R_\varphi(\underline{x}_0) \geq R_\varphi(\underline{x}))$

Find cont bounded function $\mathbb{R} \to \mathbb{R}$ that does <u>not</u> attain its max.

(or its min.)



Arctan(x).

<u>Thm.</u> If $f : [a, b] \to \mathbb{R}$ is cont, / finite closed int

then it attains its max.

How to generalize?

<u>Def</u> $S \subseteq \mathbb{R}$ is <u>closed</u> if it contains all of its limit points.

<u>Def</u> x is a <u>limit point</u> of S if

$$(\forall \epsilon > 0)([x - \epsilon, x + \epsilon] \cap S \neq \emptyset)$$

Closure of S: $\overline{S}$: set of limit pts.

$\overline{\mathbb{Q}} = \mathbb{R}$ $\quad$ $\overline{\text{diagonalizable matrices over } \mathbb{C}}$ = all matrices over $\mathbb{C}$ (density)

Thm If $S \subseteq \mathbb{R}^n$ is closed and bounded, then
$\forall f : S \to \mathbb{R}^n$ s.t. f cont, f attains its maximum.

Def A subset of a finite-dim Euclidean space is compact if it is closed and bounded

$R_\varphi(\lambda \underline{x}) = R_\varphi(\underline{x})$ (b/c num and denom both multiplied by λ^2).

$\lambda \neq 0$

{Values of R_φ over $V \setminus \{0\}$} =
{Values of R_φ on unit sphere :
$\{x \in V \mid \|x\| = 1\}$}

DO Unit sphere is closed.

Then $R_\varphi(\underline{x})$ will attain its maximum b/c unit sphere compact and $R_\varphi(x)$ cont, so
$(\exists \underline{x}_0 \neq 0)(\forall \underline{x} \neq 0)(R_\varphi(\underline{x}_0) \geq R_\varphi(\underline{x}))$. $\square$

Claim. $\underline{x}_0$ is an eigenvector

Proof $\mathcal{U} := \operatorname{span}(\underline{x}_0)$

NTS: $\mathcal{U}$ is φ-invariant

Eq. to showing: $\mathcal{U}^\perp$ is $\varphi^\perp$-invariant

i.e φ-invariant (by symm.)

$\downarrow$

i.e. $(\forall \underline{w} \in \mathcal{U}^\perp)(\underline{x}_0 \perp \varphi(\underline{w}))$.

$\underline{w} \in \mathcal{U}^\perp$ NTS: $\underline{x}_0 \perp \varphi(\underline{w})$.

Let $f(t) = R_\varphi(x_0 + tw) = \dfrac{\langle \underline{x}_0 + t\underline{w}, \varphi(\underline{x}_0 + t\underline{w}) \rangle}{\|\underline{x}_0 + t\underline{w}\|^2}$

$f : \mathbb{R} \to \mathbb{R}$

Denom: $\|\underline{x}_0\|^2 + t^2\|\underline{w}\|^2$ (by Pythag. Thm.)

Num: $\langle \underline{x}_0 + t\underline{w}, \varphi(\underline{x}_0 + t\underline{w}) \rangle$ (by props. of lin. trans.)

$= \langle \underline{x}_0, \varphi(\underline{x}_0) \rangle + \langle \underline{x}_0, t\varphi(\underline{w}) \rangle + \langle t\underline{w}, \varphi(\underline{x}_0) \rangle + \langle t\underline{w}, t\varphi(\underline{w}) \rangle$

$= \langle \underline{x}_0, \varphi(\underline{x}_0) \rangle + t\left(\langle \underline{x}_0, \varphi(\underline{w}) \rangle + \underbrace{\langle \underline{w}, \varphi(\underline{x}_0) \rangle}_{\langle \varphi(\underline{x}_0), \underline{w} \rangle}\right) + t^2 \langle \underline{w}, \varphi(\underline{w}) \rangle$

φ symm. so $\langle \underline{x}_0, \varphi(\underline{w}) \rangle = \langle \varphi(\underline{x}_0), \underline{w} \rangle$ and

$= \langle \underline{x}_0, \varphi(\underline{x}_0) \rangle + 2t \langle \underline{x}_0, \varphi(\underline{w}) \rangle + t^2 \langle \underline{w}, \varphi(\underline{w}) \rangle$.

Thus $R_\varphi(\underline{x_0} + t\underline{w}) = \dfrac{\langle \underline{x_0}, \varphi(\underline{x_0}) \rangle + 2t \langle \underline{x_0}, \varphi(\underline{w}) \rangle + t^2 \langle \underline{w}, \varphi(\underline{w}) \rangle}{\|x_0\|^2 + t^2 \|w\|}$

NTS: $\langle \underline{x_0}, \varphi(\underline{w}) \rangle = 0$.

Note $f(0) \geq f(t) \quad \forall t \in \mathbb{R}$ b/c

$$\begin{array}{ccc} f(0) & \geq & f(t) \\ \| & & \| \\ R_\varphi(\underline{x_0}) & \geq & R_\varphi(\underline{x}) \end{array}$$

where $\underline{x} = \underline{x_0} + t\underline{w}$. by def of $\underline{x_0}$.

If $\|\underline{w}\| = 0$ then $\underline{x_0} \perp \varphi(\underline{0})$ $(\varphi(\underline{0}) = \underline{0})$.

so assume $\|\underline{w}\| \neq 0$. $(\underline{w} \neq 0)$

$\|\underline{x_0}\| \neq 0$ b/c $x_0 \neq 0$ by def and $\|x_0\| \geq 0$

so by 1st DO from today $\langle \underline{x_0}, \varphi(\underline{w}) \rangle = 0$.

Then $\underline{x_0} \perp \varphi(\underline{w}) \quad \forall \underline{w} \in U^\perp$, so

$U^\perp$ is φ-invariant and U is φ-invariant

and so $\underline{x_0}$ is an eigenvector. $\square$

Proof 2. $A \in M_n(\mathbb{F})$ and $f_A(\lambda) = 0$ for some

$\lambda \in \mathbb{F}$ then char. poly

λ is an eigenvalue (over $\mathbb{F}$).

i.e. $(\exists \underline{v} \in \mathbb{F}^n)(A\underline{v} = \lambda \underline{v})$.

b/c $(\lambda I - A)$ singular. $\Rightarrow$ $(\lambda I - A)\underline{v} = \underline{0}$ has nontrivial soln. in $\mathbb{F}$.

For Lemma, NTS:

$A \in M_n(\mathbb{R})$, $A = A^T$ then $\exists$ real eigenvalue

Show: If $\lambda \in \mathbb{C}$ is a complex eigenvalue then $\lambda \in \mathbb{R}$;

$z \in \mathbb{C}$ is real $\Leftrightarrow$ $z = \bar{z}$. $\longrightarrow$ i.e. $\lambda = \bar{\lambda}$.

$A \in M_n(\mathbb{C})$ $\quad A = (a_{ij})$ $\quad A = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$

$A^* =$ conjugate transpose of A

$A^* = (b_{ij})$ where $b_{ij} = \overline{a_{ji}}$ $\quad A^* = \begin{pmatrix} \bar{z}_{11} & \bar{z}_{21} \\ \bar{z}_{12} & \bar{z}_{22} \end{pmatrix}$

"adjoint"

Def. A is Hermitian if $A = A^*$.
("self-adjoint").

Obs. $A \in M_n(\mathbb{R})$ is Hermitian $\Leftrightarrow$ A is symmetric

Thm. All eigenvalues of a Hermitian matrix are real.

DO $(AB)^* = B^* A^*$ $\quad$ ($(AB)^T = B^T A^T$ and

$\overline{zw} = \overline{z}\,\overline{w}$, $\overline{z+w} = \overline{z} + \overline{w}$,

$\therefore \overline{AB} = \overline{A}\,\overline{B}$).

Proof (of Thm.)

$A\underline{x} = \lambda \underline{x}$ where $\underline{x} \in \mathbb{C}^n$, $\underline{x} \neq \underline{0}$, $\lambda \in \mathbb{C}$.

NTS: $\lambda = \overline{\lambda}$.

Hermitian quadratic form: $f(x) = x^* A x$

$= x^* (\lambda x)$

$= \lambda x^* x$

$z = a + bi$

$\overline{z} = a - bi$

$z\overline{z} = a^2 - (bi)^2$

$= a^2 + b^2$

$= |z|^2$.

If $\underline{u} \in \mathbb{C}^n$, $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $u_i \in \mathbb{C}$

$$\underline{u}^* \underline{u} = [\overline{u}_1 \ \overline{u}_2 \ \dots \ \overline{u}_n] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \sum_{i=1}^{n} \overline{u}_i u_i$$

$$= \sum_{i=1}^{n} |u_i|^2 = \|u\|^2.$$

Then $f(x) = \lambda \|x\|^2$

and $\|x\| > 0$, real.

$f(x) \in \mathbb{C}$, so $\overline{f(x)} = (x^* A x)^* = x^* A^* x^{**}$

$\downarrow$ $\downarrow$

A x

b/c Hermitian

$= x^* A x = f(x)$.

$\overline{f(x)} = \overline{\lambda \|x\|^2} = \bar{\lambda} \cdot \overline{\|x\|^2}$ $\|x\|^2$ b/c real, so

$= \bar{\lambda} \cdot \|x\|^2$

it follows that $\lambda \cdot \|x\|^2 = \bar{\lambda} \cdot \|x\|^2$ and

$\lambda = \bar{\lambda}$ ($\|x\|^2 \neq 0$ b/c $x \neq 0$).

$\square$

So λ is real eigenvalue.

$\mathbb{R}^n$ $A = [\underline{e}_1, \dots, \underline{e}_n]$ $I_n = (\delta_{ij})_{n \times n}$

cols: $\underline{e_i}^T \underline{e_j} = \delta_{ij}$.

<u>Def</u> A is an <u>orthogonal matrix</u> if its columns are <u>orthonormal</u>

<u>3rd Miracle of Linear Algebra</u>: If A is an orthogonal matrix, its rows are orthonormal.

$O(n)$: set of $n \times n$ orthogonal matrices.

<u>Proof</u> : $A \in O(n) \iff A^T A = I$.

rows of B are ONB $\iff BB^T = I$.

<u>NTS:</u> $A^T A = I \Rightarrow AA^T = I$.

<u>Remark</u>. AA^T is symmetric b/c $(AA^T)^T = A^{TT} A^T = AA^T$.

<u>NTS:</u>
$A^T A = I \iff A^T = A^{-1}$.

① $\exists A^{-1}$ b/c cols. are ONB.

② right multiply $A^T A = I$ by A^{-1}:

$$A^T A A^{-1} = I A^{-1}$$
$$A^T I = I A^{-1}$$
$$A^T = A^{-1}.$$

Then $AA^T = AA^{-1} = I$. $\square$

we use the 2nd Miracle here... let's make it more clear

DO

R is ring with identity

$x, y, z \in R$

① $xy = 1$ x is left inverse of y

② $yz = 1$ z is right inverse of y.

Then $x = z$.

Proof

$xy = 1$ (①)

$(xy)z = 1 \cdot z$ (right multiply by z)

$x(yz) = 1 \cdot z$ (associativity)

$x \cdot 1 = 1 \cdot z$ (②)

$x = z$. (mult. identity)

or $z = \underbrace{(xy)z = x(yz)}_{\text{associativity}} = x$.

Cor. $\exists$ left inverse and $\exists$ right inverse $\Leftrightarrow$ $\exists$ inverse (two-sided)

If dim V is finite, $\varphi : V \to V$, φ has left inverse $\Rightarrow$ $\exists\, \varphi^{-1}$.

DO False in infinite dimensions – find φ with more than one left inverse.

Suppose $A \in \mathbb{F}^{k \times l}$.

Thm
A has a right inverse $\Leftrightarrow$ col $(A) = \mathbb{R}^k$ i.e. col rk$(A) = k$.

$\exists B \in \mathbb{F}^{l \times k}$ s.t. $\underset{k\times l\ l\times k}{AB} = I_{k\times k} = [\underline{e}_1 \ldots \underline{e}_k]$.

$B = [\underline{b}_1 \text{ -- } \underline{b}_k]$ (cols of B)

$A\underline{b}_j = \underline{e}_j$

$\exists\, b_j \Leftrightarrow \underline{e}_j \in \text{col}(A)$ (colum space of A)

Thm. A has left inverse $\Leftrightarrow$ row rk$(A) = l$.

Cor. $A \in M_n(\mathbb{F})$ has left inv $\Leftrightarrow$ row rk$(A) = n \Leftrightarrow$ col rk$(A) = n \Leftrightarrow$

2nd miracle.

$A \in M_n(\mathbb{F})$ has right inv.

Hadamard matrix: (H-matrix)

$A \in M_n(\pm 1)$

s.t. columns are orthogonal.

DO Show: rows are orthogonal.

DO Find an H-matrix for $n = 2^k$.

DO If $n \times n$ H-matrix exists then

$n = 2$ or $4 \mid n$.

$\boxed{\text{CH}}$ If $p \equiv -1 \pmod 4$ $\exists$ H-matrix that is

$(p+1) \times (p+1)$.

DO If $A \in O(n)$, then all (complex) eigenvalues

of A have $|\lambda| = 1$.