Laplacian:

$L_G = D_G - A_G$ — adjacency matrix.

$D_G$ = diag(deg(1), ..., deg(n))

(1) $L_G \cdot \mathbb{1} = 0$.

$\mathbb{1}$ — all 1's vector

In each row... → $v_1 \sim v_2$, $\sim v_3$, $v_6$

$[3 \; -1 \; -1 \; 0 \; 0 \; -1]$

So sum of row is 0.

Multiplying by $\mathbb{1}$ gives sum of row $\therefore 0$.

(2) $L_G$ is positive semidefinite.

In particular, $x^T L_G x = \sum_{i \sim j} (x_i - x_j)^2$

$\sim$ connected by an edge.

Directed incidence matrix $B(G)$ — $|V| \times |E|$ matrix.

rows labeled by $V$ entry $e = (v_{i_1}, v_{i_2})$

columns labeled by $E$.

+1 at row $v_{i_1}$

−1 at row $v_{i_2}$

0 elsewhere.

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**NTS:** $B(G)B(G)^T = L_G$

$$\begin{array}{cc} & E \\ V & \begin{bmatrix} \text{---} \hat{i} \text{---} \end{bmatrix} \end{array} \begin{array}{cc} & V \\ E & \begin{bmatrix} | \\ \hat{j} \\ | \end{bmatrix} \end{array}$$

$i \cdot i$: number of edges ($1 \cdot 1$ or $-1 \cdot -1$)

$\Rightarrow$ deg of $i$ along diags.

$i \cdot j$, $i \neq j$: only $-1$ if connected by same edge. (a 1 and $-1$ pair)

$\therefore B(G)B(G)^T = L_G$

Now $x^T B(G)B(G)^T x = (B(G)^T x)^T (B(G)^T x)$

$y^T y$

$\sum_{i \sim j} (x_i - x_j)^2$

$B(G)^T x$ is $x_i - x_j$ in each row, where $i$ is the location of $+1$ and $j$ is location of $-1$.

$\therefore$ $L_G$ is positive semidefinite. $\square$

$(B(G)^T x)^T (B(G)^T x)$ is dot product of each row of $(B(G)^T x)$ w/ itself.

## Gram Matrix

$v_1, \ldots, v_k \in V$

$G(v_1, \ldots, v_k) = (\langle v_i, v_j \rangle) = G$

(1) G is positive semidefinite.

$$M_{k \times n} = \begin{bmatrix} - v_1 - \\ - v_2 - \\ \vdots \\ - v_k - \end{bmatrix}$$

$MM^T = G$

(row-x transpose col = inner product)

$x^T M M^T x$

$(M^T x)^T (M^T x) \rightarrow$ by same logic as last prob, positive semidefinite.

(2) nonsingular $\Leftrightarrow v_1, \ldots, v_k$ linearly independent

In $\mathbb{R}$, $rk(A^T A) = rk(A)$, so if M has full row rank, G does too. (Note if M has less cols than rows the rows must be l.i. dep and G is singular)

$$\text{vol}_k(\text{Para}(v_1, \ldots, v_k)) = \pm \det \begin{bmatrix} - v_1 - \\ - v_2 - \\ \vdots \\ - v_k - \end{bmatrix}_{k \times k}$$

where $v_i \in \mathbb{R}^k$.

$\downarrow$ only holds for equal dim.

$$\det G = \det(M)\det(M^T)$$
$$= \det(M)^2$$

we know $\text{vol}_k(\text{Para}(v_1, \ldots, v_k)) = \pm \det M$,
$$\| \quad \sqrt{\det G}.$$

what if $v_1, \ldots, v_k \in V$? (arbitrary Euclidean space).

Pick ONB of $\text{span}(v_1, \ldots, v_k)$.

$$\tilde{M} = [v_1 \ \ldots \ v_k]_{ONB}$$

$\tilde{M}$ is $k \times k$ by 1st Miracle

If some of $v_1, \ldots, v_k$ lin. dep, vol 0. (all lower-dim subspaces have vol. 0)

Is $G(v_1, \ldots, v_k) = \tilde{M}\tilde{M}^T$?

Recall: $\langle v_i, v_j \rangle$ $(v_i \neq v_j)$

ONB $(u_1, \ldots, u_k)$ $= \sum_{m=1}^{k} (v_i \cdot u_m)(v_j \cdot u_m)$

$$v_i = \begin{bmatrix} \alpha_{i1} \\ \vdots \\ \alpha_{ik} \end{bmatrix} = [v_i]_{ONB}$$

$= \sum_{m=1}^{k} \alpha_{im}\alpha_{jm} = [v_i]_{ONB} \cdot [v_j]_{ONB}$. ✓

$A^* A = I$ $\quad A \in M_n(\mathbb{C})$ $\quad A^*$: conjugate transpose

$\hookrightarrow$ unitary

$A \in O(n)$ orthogonal matrix

$\hookrightarrow A \in M_n(\mathbb{R})$

$A^T A = I$

$x$ is a complex vector: $x \in \mathbb{C}^n$.

$A x = \lambda \underline{x}$

WTS: $|\lambda| = 1$.

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad x^* = [\overline{x_1} \ \ldots \ \overline{x_n}]$

$x^* A^* A x = (Ax)^*(Ax)$

$= (\lambda x)^*(\lambda x)$

$= \lambda^2 x^* x$

$= \lambda^2 \|x\|^2$

$x^* x = \|x\|^2$ b/c each entry is $\overline{x_i} x_i = |x_i|^2$ and $\|x\|^2 = \sum_i |x_i|^2$.

but $x^* A^* A x =$

$x^* x = \|x\|^2$

so $|\lambda|^2 \|x\|^2 = \|x\|^2$

and $|\lambda|^2 = 1$ so $\boxed{|\lambda| = 1}$. $\square$

(Note since $x$ eigenvector that $\|x\| \neq 0$).

Lecture.

Hermitian dot product in $\mathbb{C}^n$

$\underline{x}, \underline{y} \in \mathbb{C}^n$

$\underline{x} \cdot \underline{y} = \underline{x}^* \underline{y} = \sum \overline{x_i}\, y_i$

$\underline{x}^* \cdot \underline{x} = \sum |x_i|^2 > 0$ unless $\underline{x} = \underline{0}$

$= \|x\|^2$ (definition)

Note $\|x\|^2 \in \mathbb{R}$.

$\underline{x} \perp \underline{y}$ if $\underline{x}^* \underline{y} = 0$.

Hermitian space: $V$ over $\mathbb{C}$ with Hermitian inner product:

positive definite sesquilinear Hermitian inner product

$A = A^* \rightarrow$ Hermitian matrix

Thm All eigenvalues are real.

$f: V \times V \to \mathbb{C}$ is sesquilinear if
$\frac{3}{2}$

(1) linear in second variable:

(1a) $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$

(1b) $f(x, \lambda y) = \lambda f(x, y)$

(2) $\frac{1}{2}$ - linear in first variable

(2a) $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$

(2b) $f(\lambda x, y) = \bar{\lambda} f(x, y)$ ☆

Hermitian $\Rightarrow$

$f(x, y) = \overline{f(y, x)}$.

If $f$ Hermitian $\Rightarrow$ quadratic form $f(x, x)$ always real.

$f(x, x) = \overline{f(x, x)}$.

Positive definite: $f(x, x) > 0$ unless $x = 0$.

<u>Ex.</u> $f: [0, 1] \to \mathbb{C}$

$\langle f, g \rangle = \int_0^1 \bar{f} \cdot g \cdot dt$

Unitary matrix: columns are ONB of $\mathbb{C}^n$
↳ wrt Hermitian dot prod

$A^* A = I$.

$V, W$ complex Hermitian spaces. "conjugate-transpose" = "adjoint"

$\varphi : V \to W$ $\quad (\exists!)(\varphi^* : W \to V)$

s.t. $(\forall x \in V)(\forall y \in W)(\langle \varphi(x), y \rangle_W = \langle x, \varphi^*(y) \rangle_V)$

(Thm. — DO — proof sim to real.)

DO Gram-Schmidt for Hermitian spaces.

Proof $[\varphi^*]_{ONB} := [\varphi]^*_{ONB}$

Def. $\varphi$ is a unitary transformation of $V$ if $\varphi$ preserves the inner product. ~ "congruences" (preserve distance)

Real Euclidean space

$\varphi : V \to V$ is an orthogonal transformation if it preserves the inner product

i.e. $(\forall x, y \in V)(\langle x, y \rangle = \langle \varphi(x), \varphi(y) \rangle)$.

but $\langle x, y\rangle = \langle \varphi x, \varphi y \rangle = \langle x, \varphi^* \varphi y \rangle \quad \forall x, y$

so $\varphi^* \varphi = \underline{id}$ (identity transformation)

$\forall x, y \quad x^T A y = x^T B y \Rightarrow A = B$.

DO $(\forall x, y)(\langle x, \varphi y \rangle = \langle x, \psi y \rangle) \Rightarrow \varphi = \psi$.

Thus, $\varphi^* = \varphi^{-1}$. (In real: $\varphi^T = \varphi^{-1}$)

If $\varphi^* = \varphi^{-1}$ then $\forall x \quad \|\varphi x\| = \|x\|$, so

if $\varphi x = \lambda x$ then

$\|x\| = \|\varphi x\| = |\lambda| \|x\|$ so $|\lambda| = 1$.

Complex Gram matrix

$v_1, \dots, v_n \in V$ (Hermitian space)

$G(v_1, \dots, v_k) = (\langle v_i, v_j \rangle)_{k \times k}$

DO $G^* = G$ ($G$ is Hermitian), positive semidefinite,

nonsingular $\Leftrightarrow v_1, \dots, v_k$ lin. indep.

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Thm. V complex Hermitian space.

$\underline{e} = (e_1, \dots, e_n)$ ONB

$\Rightarrow \langle \underline{x}, \underline{y} \rangle = [x]_{\underline{e}}^* [y]_{\underline{e}}$

$x \in \mathbb{F}^n$ $b_1, \dots, b_n$ : basis of $\mathbb{F}^n$

$[x]_{\underline{b}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ $\underline{x} = \sum \alpha_i b_i$

$\Uparrow$

system of lin. equations.

$\mathbb{R}$ or $\mathbb{C}$

$\Uparrow$ $\Uparrow$

Euclidean Hermitian

$e_1, \dots, e_n$ ONB of V

$\underline{x} = \sum_{i=1}^{n} \alpha_i e_i$

$\boxed{\alpha_i = \langle e_i, x \rangle}$ (take dot product of $\underline{x}$ w/ any basis vector to get that coefficient.)

$\Uparrow$

coordinates wrt ONB

"Fourier coefficients"

$\alpha_i = \int f(x) \underline{\cos(2x)} dx$

$\uparrow$

ONB.

DO $f : [0, 2\pi] \to \mathbb{R}$ cont

$$\langle f, g \rangle = \int_0^{2\pi} f \cdot g \, dt$$

then $1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots$ are orthogonal

<u>Question:</u> Characterize those $\varphi : V \to V$ ($V$ is a complex Hermitian space) for which there exists an orthonormal eigenbasis.

over $\mathbb{R}$?

Suppose $A \in M_n(\mathbb{R})$ has ON eigenbasis $e_1, \ldots, e_n$.

$A e_i = \lambda_i e_i$.

$\underline{x} = \sum \alpha_i e_i$

$A \underline{x} = \sum \alpha_i \lambda_i e_i$

Eigensubspaces $U_\lambda$

$$U_\lambda = \{ x \in \mathbb{R}^n \mid A \underline{x} = \lambda \underline{x} \}$$

If $\exists$ eigenbasis then

$$\mathbb{R}^n = \bigoplus_\lambda U_\lambda$$

If $\exists$ ON eigenbasis then

$\Rightarrow U_\lambda \perp U_\mu$ if $\lambda \neq \mu$ are eigenvalues.

Orthogonal projection

$V$ is finite-dim Euclidean space.

$U \leq V$ $\quad$ $\forall v \in V$,

$U^\perp \oplus U = V \to v = u + w$ where $u \in U$ and $w \in U^\perp$.

Define linear map $\pi_U : v \mapsto u$.

$u = \pi_U(v)$. $(\pi_U : V \to U)$.

Cor. If $\varphi : V \to V$ symmetric $(\mathbb{R})$

then $\varphi = \sum_\lambda \lambda \cdot \pi_{U_\lambda}$.

Proof: $x = \sum_\lambda u_\lambda$ where $u_\lambda \in U_\lambda$ $\forall \lambda$.

so $u_\lambda = \pi_{U_\lambda}(x)$ b/c $\sum_{\mu \neq \lambda} u_\mu \perp u_\lambda$.

$\varphi x = \sum_\lambda \varphi(u_\lambda) = \sum_\lambda \lambda u_\lambda = \sum_\lambda \lambda \pi_{U_\lambda}(x)$. $\square$

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This is an equivalent form of the Spectral Thm.

If $\varphi = \varphi^T$ in a finite dim Euclidean space then $V =$ direct sum of orthogonal subspaces $U_i$ s.t. $\varphi = \sum \lambda_i \pi_{U_i}$.

DO $\pi_U^{\perp} = \pi_U$.

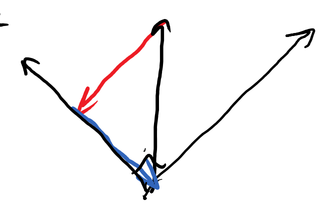
Cor. A symmetric $(\mathbb{R}) \iff \exists$ ON eigenbasis.

$\Rightarrow$ spectral Thm.

$\Leftarrow$ what we just did

$\boxed{\pi_U^2 = \pi_U}$ $\leftarrow$ idempotent $x^2 = x$

DO If $U_1 \perp U_2$, then $\pi_{U_1} \pi_{U_2} = 0$.



Eigenvalues of projections:

$\lambda^2 = \lambda$

$\lambda(\lambda - 1) = 0$

$\lambda = \{0, 1\}$.

For complex... the answer is not the same

$A \in M_n(\mathbb{C})$

Hermitian : $A^* = A$

Unitary : $A^* = A^{-1}$

<u>Def</u> $A$ is <u>normal</u> if $AA^* = A^*A$.

<u>Thm</u> (Complex Spectral Theorem)

$A$ has an orthonormal eigenbasis $\iff$ $A$ is normal.

<u>Thm</u> $\varphi : V \to V$ (Hermitian space, finite-dim)

$\Rightarrow \exists$ max. chain of invariant subspaces.

$\{0\} = U_0 < U_1 < \cdots < U_n = V$ $\quad$ $\dim U_i = i$

$\varphi(U_i) \subseteq U_i \quad \forall i \in [n]$

Eq. to saying if $A \in M_n(\mathbb{C})$, then $A \sim$ ◥.

<u>i.e.</u> $(\exists B \in M_n(\mathbb{C}))(\exists B^{-1}$ and $B^{-1}AB$ is triangular$)$

<u>Thm.</u> $\exists B \in U(n) = \{n \times n$ <u>unitary</u> matrices$\}$

↳ orthonormal basis change.

<u>Proof</u> Eq. to finding ONB $e_1, \dots, e_n$ s.t. $e_i \in U_i$.

such a basis $b_1, \dots, b_n$ exists $\to$ Gram-Schmidt + normalize. $\square$

DO If $A \in M_n(\mathbb{R})$ and all eigenvalues of $A$ are real $\Rightarrow$ $A \sim_{\text{ortho}} \blacktriangle$.

$A, B \in M_n(\mathbb{C})$.

$A$ is unitarily similar to $B$ ($A \sim_u B$) if $\exists\, S \in U(n)$ s.t. $B = S^{-1}AS = S^*AS$.

DO If $A \sim_u B$ and $A$ is normal then $B$ is normal.

DO Diag. matrices are normal.

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$D^* = \begin{pmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{pmatrix}$$

$$D^*D = \begin{pmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{pmatrix} = DD^*$$

This shows $\exists$ ON eigenbasis $\Rightarrow$ normal.

$(\Rightarrow)$

$\Leftarrow$ May assume $A$ is , why?

Every complex matrix is unitarily similar to a triangular matrix.

DO $U(n)$ is a group (closed under multiplication & inverses)

DO If a triangular matrix is normal, then it is diagonal

(Prove by induction on dim.) This shows $\Leftarrow$.

---

Spectral Theorem, restated.

If $A \in M_n(\mathbb{R})$ and $A = A^T$ then $\exists\, S \in \underbrace{O(n)}_{\text{orthogonal}}$ s.t.

$S^{-1}AS$ is diagonal

(Every symmetric real matrix is orthogonally similar to a diagonal matrix.)

# Singular Value Decomposition

$A \in \mathbb{R}^{k \times l} \Rightarrow$

$\exists\ S \in O(k),$
$T \in O(l)$ s.t. $S^{-1}AT = \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ 0 & & & 0 & \\ & & & & \ddots \ 0 \end{bmatrix}$

where $r = rk(A)$

$\sigma_1 \geq \dots \geq \sigma_r > 0$

"singular values of $A$" - unique.

DO Show SVD $\Updownarrow$ this result

Real Euclidean spaces

$\varphi : \underset{k}{V} \to \underset{l}{W}$

$\Rightarrow \exists$ ONB $e_1, \dots, e_k$ in $V$ and
ONB $f_1, \dots, f_l$ in $W$ and
$\sigma_1 \geq \dots \geq \sigma_r > 0$ unique

s.t. for $1 \leq i \leq r$ $\quad \varphi(e_i) = \sigma_i f_i$
$\varphi^T(f_i) = \sigma_i e_i$

for $j > r$ $\quad \varphi(e_j) = 0$
$\varphi^T(f_j) = 0$

Follows from Spectral Thm

$\underbrace{\varphi^T\varphi}(e_i) = \varphi^T(\sigma_i f_i) = \sigma_i^2 e_i$

$i = 1, \dots, r$ $\quad \varphi^T\varphi$ is pos. semidefinite symm.

$\lambda_1 \ge \dots \ge \lambda_n \ge 0$

$\lambda_1 \ge \dots \ge \lambda_r > 0 = \lambda_{r+1} = \dots = \lambda_n$.

so $\boxed{\sigma_i = \sqrt{\lambda_i} \text{ for } i = 1, \dots, r}$

$(e_i)$ ON eigenbasis of $\varphi^T\varphi$ DO verify this.

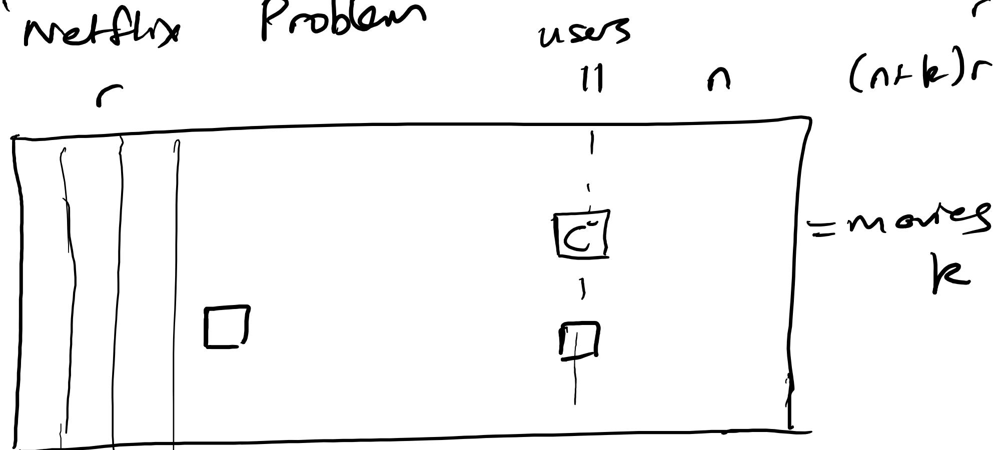
$(f_i)$ ON eigenbasis of $\varphi\varphi^T$

This has many applications in modern math —

"Netflix Problem" $\qquad A = \underset{r \text{ rows}}{B} \cdot \underset{r \text{ cols}}{C}$

$(n+k)r$

$\underline{\underline{n \cdot k}}$ dof



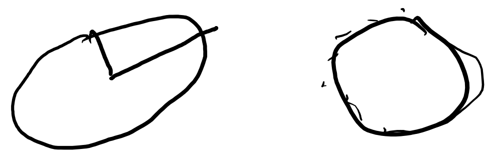

sparsely populated — Netflix wants to guess empty entries and see which movies you might like.

Suppose there existed an ideal matrix with everyone's <u>true</u> rating over every matrix.

How can we see this matrix with an imperfect observations?

Occam's Razor: simplicity = truth

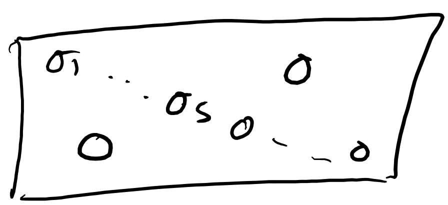
(e.g. Kepler's laws)



Note the "Netflix matrix" is low rank (relative to # of entries)

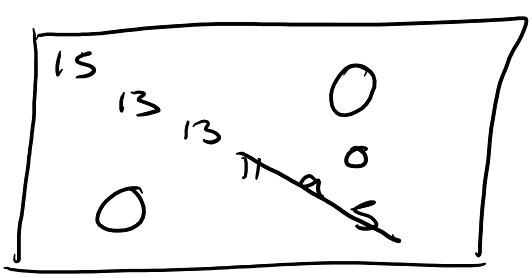
- low rank approximation

Human preferences follow low rank approx.?




$rk(A) = s$

$r < s$




transform back - closest rank r approx.