Asymptotic Notation

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1 Preliminaries

Notation: $\exp(x) = \mathrm{e}^x$.

Throughout this course we shall use the following shorthand in quantifier notation.

$(\forall a)$ is read as "for all a"

$(\exists a)$ is read as "there exists a **such that**"

$(\forall a, \text{statement}(a))$ is read as "for all a such that statement(a),..."

Example. $(\forall x \neq 0)(\exists y)(xy = 1)$ is a statement which holds in every field but does not hold e. g. in $\mathbb{Z}$.

The symbol $[n]$ will be used to denote $\{1, 2, \ldots, n\}$ in combinatorial contexts.

Definition 1.1 Let a_n be a sequence of real of complex numbers. We write $\lim_{n\to\infty} a_n = c$ (or simply $a_n \to c$) if

$$(\forall \epsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n > n_0)(|a_n - c| < \epsilon).$$

2 Asymptotic Equality

Often, we are interested in comparing the rate of growth of two functions, as inputs increase in length. Asymptotic equality is one formalization of the idea of two functions having the "same rate of growth."

Definition 2.1 We say a_n is *asymptotically equal* to b_n (denoted $a_n \sim b_n$) if $\lim_{n\to\infty} a_n/b_n = 1$.

Exercise 2.2 Consider the following statement.

$$\text{If } a_n \sim b_n \text{ and } c_n \sim d_n \text{ then } a_n + c_n \sim b_n + d_n. \tag{1}$$

1. Prove that (1) is false.
2. Prove: if $a_n, b_n, c_n, d_n > 0$ then (1) is true. *Hint.* Prove: if $a, b, c, d > 0$ and $a/b < c/d$ then $a/b < (a + c)/(b + d) < c/d$.

Exercise 2.3
1. If $f(x)$ and $g(x)$ are polynomials with respective leading terms ax^n and bx^m then $f(n)/g(n) \sim (a/b)x^{n-m}$.
2. $\sin(1/n) \sim \ln(1+1/n) \sim 1/n$.
3. $\sqrt{n^2+1} - n \sim 1/2n$.

Next we state some of the important asymptotic formulas.

Theorem 2.4 (Stirling's Formula)

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Theorem 2.5 (The Prime Number Theorem) *Let $\pi(x)$ be the number of primes less than or equal to x.*

$$\pi(x) \sim \frac{x}{\ln x},$$

where ln *denotes the natural logarithm function.*

Definition 2.6 A *partition* of a positive integer n is a representation of n as a sum of positive integers: $n = x_1 + \ldots + x_k$ where $x_1 \leq \ldots \leq x_k$. Let $p(n)$ denote the number of partitions of n.

Examples: $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$. The 5 representations of 4 are $4 = 4$; $4 = 1+3$; $4 = 2+2$; $4 = 1+1+2$; $4 = 1+1+1+1$. One of the most amazing asymptotic formulas in discrete mathematics gives the growth of $p(n)$.

Theorem 2.7 (Hardy-Ramanujan Formula)

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) \tag{2}$$

3 Little-oh notation

Definition 3.1 We say that $a_n = o(b_n)$ ("a_n is little oh of b_n") if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Exercise 3.2 Show that $a_n \sim b_n \Longleftrightarrow a_n = b_n(1 + o(1))$.

Exercise 3.3 Use the preceding exercise to give a second proof of (1) when $a_n, b_n, c_n, d_n > 0$.

4 Big-Oh, Omega, Theta notation (O, Ω, Θ)

Definition 4.1 We say that

1. $a_n = O(b_n)$ (a_n is "big oh" of b_n) if $|a_n/b_n|$ is bounded ($0/0$ counts as "bounded"), i. e.,

$$(\exists C > 0, n_0 \in \mathbb{N})(\forall n > n_0)(|a_n| \leq C|b_n|).$$

2. $a_n = \Omega(b_n)$ if $b_n = O(a_n)$, i. e., if $|b_n/a_n|$ is bounded $(\exists c > 0, n_0 \in \mathbb{N})(\forall n > n_0)(|a_n| \geq c|b_n|)$

3. $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$, i. e.,

$$(\exists C, c > 0, n_0 \in \mathbb{N})(\forall n > n_0)(c|b_n| \leq |a_n| \leq C|b_n|).$$

Exercise 4.2 Let $a_n, b_n > 0$. Show: $a_n = \Theta(b_n) \iff \ln a_n = \ln b_n + O(1)$.

Exercise 4.3 Let $a_n, b_n > 1$. Suppose $a_n = \Theta(b_n)$. Does it follow that $\ln a_n \sim \ln b_n$?

1. Show that even $\ln a_n = O(\ln b_n)$ does not follow.

2. Show that if $b_n \to \infty$ then $\ln a_n \sim \ln b_n$ follows.

Exercise 4.4 Give a very simple proof, without using Stirling's formula, that $\ln(n!) \sim n \ln n$.

Exercise$^+$ 4.5 Prove, without using the Hardy–Ramanujan formula, that

$$\ln p(n) = \Theta(\sqrt{n}).$$

Exercise$^+$ 4.6 Let $p'(n)$ denote the number of partitions of n such that all terms are primes or 1. Example: $16 = 1 + 1 + 1 + 3 + 3 + 7$. Prove:

$$\ln p'(n) = \Theta\left(\sqrt{\frac{n}{\ln n}}\right).$$